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DETECTION CAPABILITIES OF SEVERAL PHASE-PROCESSING RECEIVERS

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Naval Underwater Systems Center
New London, Connecticut

13 July 1973

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Detection Capabilities of Several Phase-Processing Receivers

ALBERT H. NUTTALL
Office of the Director of Science and Technology

13 July 1973

NAVAL UNDERWATER SYSTEMS CENTER
New London Laboratory

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DETECTION CAPABILITIES OF SEVERAL PHASE-PROCESSING RECEIVERS

Research Report

Albert H. Nuttall

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Department of the Navy

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Detection Characteristics						
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Phase-Processing Receivers						
Probability of Detection						
Probability of False Alarm						
Receiver Operating Characteristics						
Sine Wave in Gaussian Noise						

ABSTRACT

The operating characteristics of several phase-processing receivers for detection of a sine wave in Gaussian noise are deduced both analytically and by simulation and compared with optimum processors over a wide range of signal-to-noise ratios. Among the processors considered are (1) the optimum phase-processor for small signal-to-noise ratio and known sine wave frequency, (2) the maximum likelihood phase-processor for small signal-to-noise ratio and unknown sine wave frequency, (3) the phase-difference processors for known or unknown sine wave frequency, and (4) a phase processor that measures the scatter of phase samples about the best-fitting straight line. It is found that using phase samples alone is much better than using amplitude samples alone, and is not much less effective than using amplitude and phase samples together when the number of samples is large and the signal-to-noise ratio of each sample is small; loss of amplitude information causes about 1 dB degradation in many important situations.

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LIST OF ABBREVIATIONS AND SYMBOLS

CF	Characteristic function
CFAR	Constant false alarm rate
DC	Detection characteristic
FFT	Fast Fourier transform
LR	Likelihood ratio
ML	Maximum likelihood
NBF	Narrowband filter
PDF	Probability density function
ROC	Receiver operating characteristic
RV	Random variable
SNR	Signal-to-noise ratio

P_o	Signal sine wave amplitude
ψ_o	Signal sine wave phase
f_o	Signal sine wave frequency
t	Time
T	Signal duration
$s(t)$	Signal waveform
$n(t)$	Noise waveform
σ^2	Noise variance
$r(t)$	Received waveform
$\underline{n}(t)$	Noise complex envelope
$n_c(t)$, $n_s(t)$	Noise in-phase and quadrature components
t_k	Sampling instant
x_k, y_k	In-phase and quadrature samples
R_k, θ_k	Amplitude and phase samples

LIST OF ABBREVIATIONS AND SYMBOLS (Cont'd)

M	Number of samples
f_d	Unknown frequency shift
m	Voltage signal-to-noise ratio = P_0/σ
P_F	Probability of false alarm
P_D	Probability of detection
μ	Weighting power
f	Frequency
Δ	Time between samples
N	Number of frequency bins searched
i	$\sqrt{-1}$
Re	Real part
max	Maximum
ϵ	Is contained in
E	Expectation
overbar	Expectation

DETECTION CAPABILITIES OF SEVERAL PHASE-PROCESSING RECEIVERS

1. INTRODUCTION

Optimum detection of a finite-duration sine wave of known frequency in Gaussian white noise involves the use of a matched filter. The matched filter output is sampled at the appropriate time instant if phase-coherent reception is used, whereas the envelope-detected output is sampled if the received signal phase is unknown. The sample is then compared with a threshold to decide whether a signal is present or absent. Three major practical limitations that preclude the use of these processors are that either (1) the duration or location of the sine wave may not be known, (2) the medium frequency-spread may not be known (as for example in a fading medium), or (3) the bank of filters required to match unknown Doppler shifts may be too large or too expensive to build, especially for a long duration sine wave and a multibeam receiving system. Another disadvantage of such processors is that unknown noise levels require constant monitoring and threshold adjustment in order to realize a specified false alarm probability. (The losses associated with this threshold adjustment for a phase-coherent receiver are documented in reference 1.)

What is desired is a bank of fewer filters of broader bandwidth than the unknown matched filter, accompanied by an adaptive processor that combines the appropriate number of broader-filter outputs for near-optimum detection and possesses a constant false alarm rate (CFAR) capability in unknown noise levels. One possible way of partially fulfilling these requirements is based on the following observation: Even when the output signal-to-noise ratio (SNR) of a narrowband filter has dropped to the point where signal <u>amplitude</u> indications have vanished, a time history of the output <u>phase</u> samples continues to provide straight line patterns that can be discerned by a human observer. Since the observer automatically searches out and averages the appropriate length of time history from the phase chart where the frequency line is stable, he is effectively performing a function approximating coherent processing over the appropriate (but unknown a priori) time interval; this function can be interpreted as adaptive narrowband filtering. The amount of additional SNR that can be gained from this phase-processing technique depends on the received line stability, the filter bank bandwidths, the human observer's capability, and the operating point (i.e., false alarm and detection probabilities).

The advantages offered by this proposed phase-processing technique could be realized (and surpassed) using a filter bank of the appropriate width. However, since the appropriate width is not known a priori (and will probably vary with frequency, location, environment, and local conditions) the proposed suboptimum technique is definitely worthy of consideration and will be investigated in detail here to determine, quantitatively, the amount of improvement achievable in practical situations. Actually, several phase-processing techniques will be investigated and compared with optimum processing in order to determine the limits of the capabilities of each. The CFAR capability will also be discussed when appropriate.

Section 2 presents a definition of the problem to be considered and the forms of processing to be compared for use with both known and unknown sine wave frequency. Section 3 provides quantitative evaluations, obtained both analytically and by simulation, of the performance of the various processors. Section 4 contains conclusions about the relative merits of each processor. The derivations of the processing forms are presented in appendix A; useful geometrical interpretations of some of the processors, for both known and unknown sine wave frequency, are derived and discussed in appendix B; derivations of processor performance capabilities are collected in Appendix C; and the simulation procedure and program is documented in appendix D. For ease of cross referencing the appendixes are arranged such that particular processors have the same section numbers in each appendix; e. g., Processor V for known signal frequency is treated in sections A. 1. 5, B. 1. 5, and C. 1. 5.

2. PROBLEM DEFINITION

2. 1 KNOWN SIGNAL FREQUENCY

The received signal (if present) has amplitude P_0, phase ψ_0, and frequency f_0. The sine wave frequency, f_0, is assumed known in this subsection; this restriction is removed in Section 2. 2. The sine wave amplitude, P_0, is also assumed known. As will be shown, however, none of the processors considered require or use knowledge of P_0, i. e., they are uniformly most powerful with respect to P_0 (see reference 2, pp. 88-92). Therefore, this restriction can also be removed. Various assumptions about the extent of knowledge of sine wave phase, ψ_0, are made below. The received signal, s(t), is expressed mathematically as a function of time, t, as

$$s(t) = P_0 \cos (2\pi f_0 t + \psi_0), \quad t \in T, \qquad (1)$$

where T is the signal duration. This model allows for multipath in the medium, but not for fading. More precisely, any fading must be slow enough so that no significant amplitude or phase change occurs during the time interval T.

The received noise, $n(t)$, has a variance σ^2 and is stationary, narrowband, and Gaussian. Although σ^2 is assumed known, several of the phase processors will not require or use this knowledge. The received waveform, $r(t)$, is then given by

$$r(t) = \begin{Bmatrix} s(t) + n(t), & \text{signal present} \\ n(t), & \text{signal absent} \end{Bmatrix} . \tag{2}$$

The processes described in (1) and (2) are considered to be the <u>outputs</u> of a narrowband filter (NBF) of some convenient bandwidth in a receiver preprocessor. These NBF outputs are sampled at time intervals approximately equal to the inverse filter bandwidth, so that the in-phase and quadrature output noise component samples are statistically independent. (If the received tone is not centered in the NBF, P_o is smaller than it would have been for a centered filter; thus, mismatch of filter center frequency is easily incorporated in the size of P_o. Of course, for known signal frequency, this mismatch can be avoided.)

The received waveform (for signal present) can be represented in complex envelope notation (see reference 3, chapters I and II or reference 4, appendix A) as

$$r(t) = \mathrm{Re}\left\{\exp(i2\pi f_o t)\left[P_o \exp(i\,\psi_o) + \underline{n}(t)\right]\right\} , \tag{3}$$

where f_o is also used for the noise center frequency, and noise complex envelope $\underline{n}(t)$ is given in terms of its real in-phase and quadrature components according to

$$\underline{n}(t) = n_c(t) + i\,n_s(t) . \tag{4}$$

A sample of the complex envelope of the received waveform at time t_k is therefore given by

$$\begin{aligned} P_o \exp(i\,\psi_o) + \underline{n}(t_k) &= P_o \cos(\psi_o) + n_c(t_k) + i\left[P_o \sin(\psi_o) + n_s(t_k)\right] \\ &\equiv x_k + i\,y_k \equiv R_k \exp(i\,\theta_k) , \end{aligned} \tag{5}$$

where x_k is the k-th in-phase sample and y_k is the k-th quadrature sample of the received waveform. Alternately, R_k is the k-th amplitude sample and θ_k is the k-th phase sample. For M samples available, $1 \le k \le M$, the problem we will address is how $\{R_k\}_1^M$ or $\{\theta_k\}_1^M$ or both should be processed for near-optimum detection capability when different degrees of knowledge of signal phase ψ_o and frequency f_c are available, and, in some cases, for small SNR.

The complex sample $x_k + iy_k$ can also be considered to be the output of the particular frequency bin at f_0 of a fast Fourier transform (FFT) applied directly to the received waveform prior to any prefiltering. The time between samples is then the time interval over which the FFT is conducted. The following investigation is concerned with the way in which the sequence of FFT outputs should be combined and how well the various combinations perform.

Seven processors that will be investigated for their detection capabilities are listed in table 1. The decision variable for each is compared with a threshold. If the threshold is exceeded, a signal is declared present; otherwise it is declared absent. (For Processor VI, the reverse comparison is made.) The derivations of the processors in table 1 are given in appendix A, along with the assumptions necessary to make each valid or optimal.

Table 1. Seven Signal Processors for Known Signal Frequency

Processor	Knowledge Assumed About Signal Phase ψ_o	Samples Used	Decision Variable
I	Known	Amplitudes R_k Phases θ_k	$\mathrm{Re}\left\{e^{-i\psi_o} \sum_{k=1}^{M} R_k e^{-i\theta_k}\right\}$
II	Constant but Unknown	Amplitudes R_k Phases θ_k	$\left\lvert \sum_{k=1}^{M} R_k e^{i\theta_k} \right\rvert$
III	Independent Phases	Amplitudes R_k	$\sum_{k=1}^{M} R_k^2$
IV	Known	Phases θ_k	$\mathrm{Re}\left\{e^{-i\psi_o} \sum_{k=1}^{M} e^{i\theta_k}\right\}$
V	Constant but Unknown	Phases θ_k	$\left\lvert \sum_{k=1}^{M} e^{i\theta_k} \right\rvert$
VI	Constant but Unknown	Phases θ_k	$\frac{1}{M} \sum_{k=1}^{M} \theta_k^2 - \left(\frac{1}{M} \sum_{k=1}^{M} \theta_k\right)^2$
VII	Constant but Unknown	Phases θ_k	$\mathrm{Re}\left\{\sum_{k=2}^{M} \exp(i\theta_k - i\theta_{k-1})\right\}$

The first processor in table 1 assumes knowledge of signal phase ψ_0 and uses both amplitude and phase samples $\{R_k\}$, $\{\theta_k\}$ in its decision-making procedure. It is the optimum phase-coherent processor; no other is as effective using the available samples. A geometrical interpretation of its decision variable is that an M-step random walk in the complex plane is taken, the k-th step is of length R_k and angle θ_k, and the resultant sum is rotated by the known amount, $-\psi_0$ radians, before its real projection is compared with a threshold.

Processor II does not assume knowledge of signal phase ψ_0. In this case the length of the total random walk is compared with a threshold. It is seen that if $\{\theta_k\}$ tend to cluster around a particular value (ψ_0), the length of the total walk will be large and signal detectability will be enhanced. It is shown in appendix A that Processor II is optimum when the a priori probability density function (PDF) of signal phase is uniform over 2π radians. Also, Processor II results when the maximum likelihood (ML) estimate of signal phase is used, instead, in a generalized likelihood ratio (LR) test. Thus two different approaches to the treatment of signal phase yield the same processor.

In case III we assume that signal phase is independent every sample. The optimum processor in this case is the $\ln I_0$ processor (see appendix A or reference 2). An approximation to this processor for small SNR is afforded by the decision variable in table 1; it discards all phase sample information. An advantage of the approximation is that the decision variable is independent of P_0, whereas the $\ln I_0$ processor requires knowledge of both P_0 and σ for its realization. Thus the approximation is uniformly most powerful with respect to P_0.

Both $\{R_k\}$ and $\{\theta_k\}$ were available for use by Processors I, II, and III for every sample instant. Since increased noise levels cause an increase in the value of the decision variable in these processors, it is readily apparent that none of them can possibly be a CFAR receiver. In contrast, the last four processors shown in the table are restricted to operating on phase samples only, and will possess CFAR capability for a fixed known value of M, the number of samples. For example, Processor IV-assumes knowledge of signal phase ψ_0, and its decision variable is identical to that of Processor I, except that the length of each individual random walk is equal to unity. It is shown in appendix A that, for small SNR, Processor IV is the optimum processor operating solely on phase samples; i. e., no other processor restricted to employing only $\{\theta_k\}$ can do better for small SNR.

Processor V does not require that ψ_0 be known, but does require that ψ_0 be constant during all M samples. The resultant processor is identical to Processor II except that the length of each individual random walk is again restricted to unity. It is shown in appendix A that Processor V is optimum when the a priori PDF of the signal phase is uniform, the SNR is small, and only phase samples are available for decision-making. Also, Processor V results when the SNR is small and the ML estimate of signal phase is used, instead, in a generalized LR test.

Processor VI attempts to characterize, mathematically, what happens when a human observes a phase-versus-time chart constructed from $\{\theta_k\}$. Specifically, the best horizontal line is fitted to the phase-time chart (for known signal frequency f_0), such that the average squared-error over M samples is minimized. The remaining scatter about the best straight line is used as a decision variable (see appendix A for derivations). The samples need not be equi-spaced in time. There is an ambiguity inherent in Processor VI because of the nonuniqueness of phases within multiples of 2π; this problem and methods of circumventing it are discussed in section 3.

Processor VII is a phase-difference processor that is not optimal in any sense. It is, however, a practical compromise that can be resorted to if the phase of the signal is expected to make an abrupt change somewhere in the middle of the sequence of M samples, or if the phase drifts significantly over the total sequence length. An abrupt phase change drastically deteriorates the performance of all the other processors in table 1 (except Envelope-Processor III) because the coherent addition can become destructive rather than constructive. However, for Processor VII, an abrupt phase change merely causes several noisy samples in the coherent addition. Also, the samples need not be equi-spaced in time. Processor VII has disadvantages that will become evident when its detection capability is investigated in section 3.

The derivations for Processors I through V that are presented in appendix A assume that the additive noise is Gaussian and that the samples are statistically independent. However, these processors can be derived without either of the restrictive assumptions, namely, by derivations that are based on a geometric interpretation of the envelope and phase samples as presented in appendix B.

Derivations of the detection and false alarm probabilities for the seven processors are given in appendix C. Where possible, exact results, or a method capable of leading to an exact result, are presented. In some cases, only the false alarm probability can be calculated exactly. In other cases, approximations to the detection and false alarm probabilities are given. For Processor VII, a more general technique than is indicated in table 1 is also investigated (see (C-86)).

2.2 UNKNOWN SIGNAL FREQUENCY

For unknown signal frequency, the performance of all the processors in table 1 (except for Processor III*) will be degraded. The received signal

*Actually, as the unknown frequency shift of the signal is made large enough to decrease the signal output of the NBF in the receiver preprocessor, P_0 decreases and performance degrades. However, our expressions for detectability will be in terms of P_0 itself, rather than in terms of the signal amplitude in the medium, the NBF transfer-function, and the precise frequency shift.

waveform is now

$$s(t) = P_o \cos(2\pi(f_o + f_d)t + \psi_o), \; t \in T, \tag{6}$$

where f_d is the unknown (Doppler) shift. Although we again assume that P_o is known, it will not be needed in any of the tests to be considered. Since it is unrealistic to assume that signal phase ψ_o would be known at some time instant, while the frequency f_d is unknown, there are no analogs of Processors I and IV here. Also, since the performance of Processor III is not affected* by a frequency shift, we need not reconsider it here.

The four processors of interest here are listed in table 2 and their derivations are presented in appendix A. If the decision variable exceeds a threshold, a signal is declared present; otherwise the signal is declared absent. (For Processor VI, the reverse comparison is made.)

Table 2. Four Signal Processors for Unknown Signal Frequency

Processor	Knowledge Assumed About Signal Phase ψ_o	Samples Used	Decision Variable
II	Constant but Unknown	Amplitudes R_k Phases θ_k	$\max_f \left\| \sum_{k=1}^{M} R_k \exp(i\theta_k - i2\pi f t_k) \right\|$
V	Constant but Unknown	Phases θ_k	$\max_f \left\| \sum_{k=1}^{M} \exp(i\theta_k - i2\pi f t_k) \right\|$
VI	Constant but Unknown	Phases θ_k	$\frac{1}{M}\sum_{k=1}^{M}\theta_k^2 - \left(\frac{1}{M}\sum_{k=1}^{M}\theta_k\right)^2 - \frac{12}{M^2-1}\left(\frac{1}{M}\sum_{k=1}^{M}\left(k - \frac{M+1}{2}\right)\theta_k\right)^2$
VII	Constant but Unknown	Phases θ_k	$\left\| \sum_{k=2}^{M} \exp(i\theta_k - i\theta_{k-1}) \right\|$

For Processor II, the complex random walk $R_k \exp(i\theta_k)$ is "unwound" by all possible rates of rotating vectors (see table 2), and the largest distance walk is compared with a threshold. It is shown in appendix A that if the a priori PDF of signal phase is uniform and the ML estimate of signal frequency is employed, the resultant test is as given in table 2. Alternatively, the appendix shows that if ML estimates are used for both signal phase and frequency, the resultant generalized LR test is identical. Thus two approaches to signal phase yield the same processor, and assumptions of small SNR are not required.

*See the footnote on page 6.

The physical interpretation of the decision variable for Processor V in table 2 is identical to that for Processor II except that the length of each component of the random walk is forced to be unity. It is shown in appendix A that this processor results for small SNR when either (1) the a priori PDF of signal phase is uniform and the ML estimate of signal frequency is used or (2) ML estimates of both signal phase and frequency are used.

Processor VI results when the best straight line is fitted to equi-spaced phase samples, such that the average squared-error over M samples is minimized. It is felt that this test approximates the way a human observer somehow pieces together the "barber pole" effect caused by an unknown frequency when confronted with a phase chart history limited to the $(-\pi, \pi)$ range. In practice, there is a real problem when attempting to apply the decision variable in table 2 to automatic data processing, because a computed phase sample θ_k will lie in the $(-\pi, \pi)$ range unless a special algorithm is used to "unwind" the phase samples along a straight line. Also, unless there is human intervention, all algorithms will occassionally yield permanent spurious $\pm 2\pi$ jumps in phase that give an unnecessarily large value to the decision variable (scatter) and can lead to erroneous signal-absent statements. If the $\{\theta_k\}$ could be unwound properly, the performance of Processor VI for unknown signal frequency would be identical to that of Processor VI for known signal frequency and, therefore, independent* of the exact frequency shift, f_d. Accordingly, no detailed operating characteristics will be given for this processor in section 3.

Processor VII is a phase-difference processor. For equi-spaced samples and unknown signal frequency, the complex vector exp $(i\theta_k - i\theta_{k-1})$ tends to cluster around an unknown phase for large SNR. The length of the total walk is used as a decision variable, since the direction of walk is not known a priori. The performance of Processor VII is independent* of the exact frequency shift f_d, which is canceled out of the decision variable (see (A-36) through (A-40)). However, the performance is poorer than that for the comparable processor for known signal frequency. The performance capability of Processor VII (see table 2) is given in section 3.

The derivations in appendix A for Processors II and V (see table 2) assume that the additive noise is Gaussian and that the samples are statistically independent. The resulting processors are derived in appendix B without either of these restrictive assumptions via a geometrical interpretation.

According to the above discussion, performance results for Processors II, V, and VII in table 2 will be presented in section 3. Analytical derivations of detection and false alarm probabilities for Processors II and V are too difficult and have not been attempted. Analytical results for the performance capability of Processor VII are presented in appendix C, where, in fact, a processor of a more general form is investigated (see (C-133)).

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*See the footnote on page 6.

3. RESULTS

3.1 KNOWN SIGNAL FREQUENCY

Receiver operating characteristics (ROCs) for the processors discussed in the previous section were determined by computer simulation using 10,000 independent trials. Although the ROCs for Processors I, II, and III are available in the literature (e.g., in references 3 and 5), they are included in this simulation for purposes of determining the credibility and accuracy of the simulation approach. Details of the simulation used are presented in appendix D.

In the following discussion, the number of samples, M, is fixed at 25 and comparisons between processors are made on this basis. For smaller values of M of interest, the program in appendix D can be suitably modified. For larger values of M, the computer time and storage increase and the analytical derivations in appendix C can be used, instead; in particular, the approximations in appendix C improve for large M.

In figure 1A, the ROC for Processor I is presented for values of

$$m \equiv \frac{P_o}{\sigma} \tag{7}$$

ranging from 0 to .8. The SNR corresponding to (7) is, in decibels,

$$10 \log \left(\frac{P_o^2/2}{\sigma^2} \right) = 20 \log (m) - 3 . \tag{8}$$

This is the SNR at the preprocessor output, prior to recombination according to table 1. In the following, P_F is the false alarm probability and P_D is the detection probability.

The (0, .1) range for false alarm probability, P_F, is expanded in figure 1B to clearly show the ROC for small P_F in the neighborhood of .01. As a check of the accuracy of the simulation approach, several exact values of performance have been added, as given by (C-5) in appendix C, and are denoted by Xs in figure 1. They are seen to virtually coincide with the simulated results.

The ROC for Processor II is presented in figure 2. Again, several exact values (from (C-9)) have been entered as Xs, and the agreement is very good over a wide range of values of P_F. In comparison with Processor I, the degree of degradation suffered by Processor II, which lacks knowledge of signal phase ψ_o, depends on the exact operating point on the ROC. For example, the point $(P_F, P_D) = (.01, .2)$ requires $m = .3$ for Processor I, but $m = .4$ for Processor II, which is a 20 log (.4/.3) = 2.5 dB degradation. Processor II

suffers a certain amount of small-signal suppression; i. e., detection of weak signals ($m << 1$) is significantly impaired in comparison with Processor I. This is characteristic of all processors that do not have knowledge of signal phase ψ_0.

The ROC for Processor III is given in figure 3. It is seen that it suffers even greater small-signal suppression; e. g., a value of $m = .78$ is required to realize the (.01, .2) operating point, which is 8.3 dB poorer than Processor I. As a comparison with figure 3, the ROC for the optimum $\ell n\, I_0$ processor, (A-16), was also computed for the same set of parameters; the results were imperceptibly different from figure 3 and, therefore, are not presented. Extensive numerical results for the performance of Processor III are available in reference 5.

The ROC for the first of the phase processors, IV, is given in figure 4. Its performance is slightly poorer than Processor I, which used both amplitude and phase samples. For example the (.1, .9) operating point is realized with $m = .52$ for Processor I, but requires $m = .58$ for Processor IV; this is a difference of only .95 dB. It is shown in appendix C (equations (C-40) and (C-41) et seq.) that for large M and small m, Processor IV is $10 \log (4/\pi) = 1.05$ dB poorer than Processor I. Thus loss of amplitude information does not cause a great loss in detection capability; rather, the information in the phase samples is the dominant contributing factor to signal detection. Since Processor IV uses knowledge of signal phase ψ_0, there is no small-signal suppression; this applies even though $\{R_k\}$ have been discarded.

Processor V does not assume knowledge of signal phase ψ_0, whereas Processor IV does. The ROC for V is depicted in figure 5. When figures 2 and 5 are compared, it is apparent that Processor V's performance capability is very similar to that of II, which had amplitude information available in addition to the phase information. For example, operating point (.1, .9) requires $m = .65$ for Processor II and $m = .74$ for Processor V — a difference of 1.1 dB. Equations (C-68) and (C-69) et seq. show that for large M and small m, Processor V is 1.05 dB poorer than Processor II. Thus, once again, loss of amplitude information causes a loss of 1.05 dB in detectability in one limiting case. Notice that Processors I, II, IV, and V all use the phase samples in the same basic way, i. e., according to the addition of complex vectors $\exp(i\theta_k)$.

Processor VI makes an entirely different use of the phase samples, and, because it does, the inherent ambiguity in phase causes some problems in automatic processing. To illustrate the problem, suppose the true signal phase $\psi_0 = 0$ and $\{\theta_k\}$ are defined in the $(-\pi, \pi)$ band. Then, for the decision variable

in table 1, the corresponding ROC for $m = 0\,(.1)\,.4$ is presented in figure 6A. This ROC is poorer* than that in figure 4A; this result agrees with the fact that Processor IV is the optimum processor for small m, phase samples alone, and known signal phase. On the other hand, the ROC in figure 6A is noticeably better than that in figure 5A because Processor VI assumed that $\psi_o = 0$, whereas Processor V does not use signal phase knowledge (although it does use $\{\theta_k\}$ optimally for small m, otherwise). Notice the absence of small-signal suppression in figure 6A, since ψ_o is known and used.

In an attempt to determine the effect of different signal phases, ψ_o, on the performance of Processor VI, the decision rule in table 1 was applied in the case where $\psi_o = \pi/2$ and $\{\theta_k\}$ were, again, in the $(-\pi, \pi)$ band. The resulting ROC in figure 6B suffers notable degradation because $\{\theta_k\}$ with values in the $(-\pi, -\pi/2)$ band heavily bias the average phase estimate, (A-33), away from the true value $\psi_o = \pi/2$. Then the scatter (see table 1 or (A-34)) about the average phase estimate is large and decisions made about signal absence are frequently erroneous. Yet, $\{\theta_k\}$ near $-\pi$ should not be allowed to contribute such a large effect since, in reality, each value $\theta_k + 2\pi$ is quite near the true value $\pi/2$; i.e., the $(-\pi/2, 3\pi/2)$ band would be the ideal interval in which to define each θ_k when $\psi_o = \pi/2$. But since the true signal phase will not be known in practice, the band to choose a priori is unknown. The problem becomes magnified when ψ_o is allowed to sweep out a full 2π range in the 10,000 trials and band $(-\pi, \pi)$ is retained for $\{\theta_k\}$. The resultant ROC in figure 6C indicates that, for certain ranges of P_F, the processor is poorer than a random choice.

A possible solution to the automatic processing problem here is to define each θ_k, simultaneously in several 2π bands, and compute the scatter that has a minimum value (in all bands considered) as the decision variable. Ideally, there should be a continuum of bands $(-\pi + v, \pi + v)$, where v ranges continuously over a 2π interval, but, since this is impractical, a few well-chosen values for the bands should be considered. In figure 6D, bands $(-\pi, \pi)$ and $(0, 2\pi)$ were both used as ψ_o swept out a full 2π range in the 10,000 trials. The improvement afforded by the addition of band $(0, 2\pi)$ is marked (compare figures 6D and 6C). In fact, for small m, the ROC in figure 6D is almost as good as that in figure 5A for the optimum phase processor without knowledge of signal phase

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*In (C-83) et seq. it is shown that Processor VI is .34 dB poorer than Processor IV for large M and small SNR m. This figure of .34 dB need not hold true over the entire ROC, however; in fact, since Processor IV is optimum only for small m, Processor VI with known ψ_o could conceivably outperform Processor IV for larger m.

ψ_0. Probably, use of the three bands $(-5\pi/3, \pi/3)$, $(-\pi, \pi)$, and $(-\pi/3, 5\pi/3)$, with centers at $-2\pi/3$, 0, and $2\pi/3$, respectively, would produce a ROC that is very close to that in figure 5A; however, this is not pursued here.

Another possible solution attempted was to "unwind" the successive phase samples $\theta_1, \theta_2, \ldots, \theta_M$ such that $|\theta_k - \theta_{k-1}| \leq \pi$; i.e., the phase samples were allowed to drift to whatever value that was necessary, provided that adjacent phase changes were never more than π radians. Then, the best tilted straight line was fitted through the data and the remaining scatter was used as a decision variable (see (A-62) et seq.); the resulting ROC in figure 6E is totally inadequate.

The above discussion has dealt with an attempt to employ automatic data processing for variations of Processor VI. If, instead, a human observes a phase chart of $\theta_1, \theta_2, \ldots, \theta_M$ versus time, limited, perhaps, to the $(-\pi, \pi)$ band, he will probably piece together the "barber pole" effect or notice clusters of phase samples about some constant nonzero value, and will be able to make accurate decisions about signal presence, even when ψ_0 is near $\pm\pi$. (Two clusters would be observed for ψ_0 near $+\pi$ or for ψ_0 near $-\pi$.) It is not known how close the human can come to the optimum performance indicated in figure 5A; however, the similarity between figures 5A and 6D for a very simple decision rule and two bands, leads to the conjecture that human performance should be fairly close to that shown in figure 5A.

A phase-difference processor for known signal frequency, which is a more general form of Processor VII than is shown in table 2, will be considered here. Specifically, the decision variable for this processor is a weighted sum of phase-difference vectors:

$$\mathrm{Re}\left\{\sum_{k=2}^{M} R_k^{\mu} \; R_{k-1}^{\mu} \; \exp\left(i\theta_k - i\theta_{k-1}\right)\right\}. \tag{9}$$

The reason for the weighting power μ is that, as has been observed above, amplitude weighting does improve performance somewhat (compare I versus IV and II versus V). Both exact and approximate detailed analyses of (9) in terms of P_D and P_F are presented in appendix C for general μ.

The ROCs for $\mu = 0$, .5, and 1, are presented in figures 7A, 7B, and 7C, respectively. These figures show improved performance as μ increases from 0 to 1. In fact, in appendix C ((C-131) et seq.) it is shown that the best value of μ, for large M and small m, is unity. Also, the loss in detectability at $\mu = 0$ versus $\mu = 1$ for Processor VII is 1.05 dB under these conditions; once again,

two processors that differ only in the way they use amplitude information differ by 1.05 dB in performance for large M and small m.

It must also be noted in figure 7 that the increment in m (= P_0/σ) is .2, not .1 as in all of the previous figures. Thus, the phase-difference processor (for any weighting μ) is obviously poorer in performance than the optimum phase processor, V, which is without knowledge of signal phase ψ_0. In fact, (C-131) shows the familiar m^2-dependence of P_D; thus, the phase-difference processor suffers a small-signal suppression effect. For larger m, a direct comparison of ROCs is necessary. For example, although the operating point (.1, .78) requires m = 1 for Processor VII with $\mu = 0$, Processor V requires only m = .63 — a difference of 4 dB. On the other hand, as was noted in the Introduction, the phase-difference processor can tolerate an abrupt signal-phase jump without disastrous effects, whereas the other processors would all be very adversely affected.

Another way of presenting the operating characteristics of a processor, which can be particularly useful for comparison of systems for small values of P_F, is to plot P_D versus m on probability paper, with P_F as a parameter (e.g., see reference 3, figures IV.1 and V.2). We shall call these plots detection characteristics (DC), in order to distinguish them from ROCs, and present them for several of the processors, as determined by both simulation and analytical methods. This approach will afford corroboration of the analytical results, as well as an indication of the adequacy and accuracy of large-M approximations. It will then be possible to use the analytical results for other values of M with assurance of their accuracy and applicability.

The DCs for Processors I, II, and III are available in references 3 and 5. The DC for Processor IV is presented in figure 8, where the solid curve represents the approximate analytical result in (C-40), with exact threshold values as given in table C-1. The superposed Xs denote the results of the simulation described above. The analytical result is expected to be in greater error* for small m than for large m. The agreement with the simulation results is quite good; the standard deviations of the results are considered in appendix D. For small P_F, approximation (C-40) is not as accurate, especially for small m; in fact, the limiting value $P_F = 10^{-4}$ is not realized at m = 0 by this approximation, even though exact thresholds were used. The simulation results for $P_F = 10^{-4}$ are not shown on figure 9 because 10,000 trials do not constitute an adequate base for estimation.

*The reason for this behavior is that the individual random variables in (C-19) are more nearly Gaussian for large m than for small m. In fact, as $m \to 0$ in (C-21), the PDF of s has cusps at ± 1.

The DC for Processor V is given in figure 9, where the analytical result, (C-63), is drawn as a solid curve; exact threshold values given by table C-2 were used. The superposed Xs again denote simulation results and it can be seen that disagreement is greatest for small P_F and small m; here the approximation gives an overestimate of attainable performance. The characteristic small-signal suppression of processors operating without knowledge of signal phase ψ_0, as exhibited by the horizontal slope at $m = 0$, is apparent.

A detection characteristic is not presented for Processor VI because, as described above, multiple bands were not attempted for the definition of $\{\theta_k\}$. However, it is anticipated that, for the reasons discussed above, the DC in figure 9 would be a good approximation to that for Processor VI.

Three DCs for Processor VII for weighting constant $\mu = 0$, .5, and 1, respectively, are presented in figure 10 (see (9)). The thresholds for $\mu = 0$ and $\mu = 1$, from tables C-3 and C-4, are exact, and the thresholds for $\mu = .5$ are determined from (C-127). The approximate P_D, as determined from (C-129), agrees well with the simulation results for all three cases of weighting constant μ considered, except in figure 10C for $\mu = 1$ and small P_F. The analytical results in figure 10C for $P_F = 10^{-3}$ and 10^{-4} are pessimistic estimates of performance for small m, but are apparently optimistic estimates for larger m. The reason for the poorer quality of the approximation to P_D for small P_F in figure 10C is that the summation random variables in (9) are far from Gaussian for $\mu = 1$ and large amplitudes. On the other hand, the extremely good agreement indicated in figure 10B is fortuitous; of course, the reason the approximation (C-129) is perfect at $m = 0$ is that approximate thresholds, as determined by (C-129) itself, have been used.

3.2 UNKNOWN SIGNAL FREQUENCY

The ROCs for Processors II, V, and VII in table 2 will be given, in keeping with the discussion in section 2. Again, 10,000 independent trials are taken and $M = 25$.

Suppose the complex samples $\{R_k \exp(i\theta_k)\}$ are taken every Δ seconds; i.e., suppose $t_k = k\Delta$ in table 2. This time interval corresponds to an NBF of width $1/\Delta$ Hz in the receiver preprocessor (or to the time between FFT outputs), as described in section 2. Thus the complete preprocessor might consist of a bank of NBFs (or a subset of the FFT outputs) extending over some selected frequency band. For a collection of M samples, the total observation interval is $M\Delta$ seconds; the fundamental frequency resolution possible over this total time

interval is $(M\Delta)^{-1}$ Hz. Therefore, if we expect the decision rules in table 2 to yield near-optimum results, the search in f must be conducted to within at least $(2M\Delta)^{-1}$ Hz.

Let the actual frequency search increment be $(N\Delta)^{-1}$, where $N \geq 2M$. Then for $f = f_n \equiv n/(N\Delta)$, the decision rule for Processor II in table 2 takes the approximate form

$$\max_n \left| \sum_{k=1}^{M} R_k \exp(i\theta_k) \exp(-i2\pi kn/N) \right| , \tag{10}$$

where the search over n covers only the range of expected frequencies. The two particular cases we consider are (1) a "track" situation where the signal frequency is known almost exactly and (2) a "search" situation where the frequency uncertainty is approximately $1/\Delta$ Hz. (For greater uncertainties, another NBF in the preprocessor bank, or a different frequency bin in the FFT output, would pick up the signal.) Mathematically, these two cases correspond to $|n| \leq 1$ and $|n| \leq N/2$, respectively, in (10).

The summation in (10) is immediately recognized as an N-point FFT of the M nonzero complex samples $\{R_k \exp(i\theta_k)\}$. Thus, (10) dictates a comparison of the maximum magnitude of the FFT (of the time sequence $\{R_k \exp(i\theta_k)\}_1^M$) with a threshold for decisions about signal presence. As N becomes very large, (10) approaches the continuous rule quoted in table 2. However, computation of (10) is unnecessarily time-consuming if N is selected too large; also, performance capability saturates for $N > 2M$. We will consider $N = 64$ here, because of the speed of the FFT for powers of 2.

The ROC for Processor II in the track situation is depicted in figure 11; it is slightly poorer than in figure 2, which shows the corresponding ROC for known signal frequency. The ROC for Processor II in the search mode is illustrated in figure 12. At operating point (.01, .62), for example, approximately 1.8 dB more SNR is required in the search mode than in the track mode. The difference is much greater for smaller m, where the lack of knowledge of signal frequency causes a significant degradation in performance.

For Processor V in table 2, the only change in (10) is to replace R_k by unity; then all of the above comments apply directly. The resultant ROC is displayed in figure 13 for the track mode and in figure 14 for the search mode. Comparison of figure 13 (unknown signal frequency) with figure 5 (known signal frequency) reveals a slight degradation. The degrading effect of the wider search procedure is evident in figure 14.

A phase-difference processor for unknown signal frequency, which is a more general form of Processor VII than is shown in table 2, will be considered here. Specifically, the decision variable is a weighted sum of phase-difference vectors:

$$\left| \sum_{k=2}^{M} R_k^{\mu} \, R_{k-1}^{\mu} \, \exp\left(i\theta_k - i\theta_{k-1}\right) \right| . \tag{11}$$

Both exact and approximate detailed analyses of (11) in terms of P_D and P_F are presented in appendix C for general μ.

The ROCs for $\mu = 0$, .5, and 1, are presented in figures 15A, 15B, and 15C, respectively. These figures show improved performance as μ increases from 0 to 1. Equation (C-177) and table C-5 in appendix C show that the best value of μ, for large M and small m, is unity. Also, the loss in detectability at $\mu = 0$ versus $\mu = 1$ is 1.05 dB under these conditions; i.e., when amplitude information is discarded in this processor, 1.05 dB is lost for large M and small m. The ROC for $\mu = 1.5$ was also computed and was found to be virtually identical to that for $\mu = 1$; however, the ROC for $\mu = 2$ was slightly poorer than the ROC for $\mu = 1$. (Neither of these cases is presented in this report.)

It is important to notice that the increment in m shown in figure 15 is .2, not .1. Thus the performance of the phase-difference processor, VII, for unknown signal frequency is extremely poor until m approaches unity; i.e., the SNR at the preprocessor output must be approximately -3 dB before adequate performance obtains (see (8)). (The degradation of Processor VII for known signal frequency (see figure 7) was not as marked for small m.) In (C-177), the approximate detection probability is shown to have an m^2-dependence rather than the m-dependence of Processors II and V. Thus, in addition to the small-signal suppression effect caused by the lack of knowledge of signal phase ψ_o, the phase-difference processor for unknown signal frequency suffers further degradation for small m for all values of weighting constant μ in (11).

Since we were unable to derive analytic results for the detection probabilities of Processors II and V for unknown signal frequency, no DCs are presented for these two cases. However, we are able to analyze Processor VII for unknown frequency; the resultant DCs for $\mu = 0$, .5, and 1 are presented in figures 16A, 16B, and 16C, respectively. The thresholds used for $\mu = 0$ and $\mu = 1$ are exact and are given in tables C-6 and C-7, respectively. The thresholds used for $\mu = .5$ were determined from approximation (C-173).

The pronounced flatness of the DCs for m near zero is expected in accordance with the suppression effect discussed above. Thus, for example, even at $m = .5$, the detection probability in figure 16A has only reached .15 for the curve labeled $P_F = 10^{-1}$. This behavior is exhibited independently of the value of μ.

The poorer quality of the approximation to P_D in figure 16C for small P_F is the result of the fact that the summation random variables in (11) are far from Gaussian for $\mu = 1$ and large amplitudes. The analytic results are pessimistic estimates of performance for small m, but optimistic for larger m.

4. DISCUSSION

The failure to use amplitude information to detect a sine wave in noise causes a degradation in performance of about 1 dB for small SNR and large M if the phase information is properly used. This conclusion is based upon a comparison of the following four pairs of processors:

a. Processor I, known signal frequency, versus Processor IV, known signal frequency;

b. Processor II, known signal frequency, versus Processor V, known signal frequency;

c. Processor VII, known signal frequency, $\mu = 1$ versus $\mu = 0$; and

d. Processor VII, unknown signal frequency, $\mu = 1$ versus $\mu = 0$.

On the other hand, discarding phase information can lead to severe degradation, as the ROC for Processor III attests.

All processors that lack knowledge of signal phase suffer small-signal suppression, regardless of their use of amplitude and phase samples. However, for larger SNR, such that useful detection probabilities result, the degradation is often insignificant. The suppression effect is more pronounced for the phase-difference processors and is, in fact, severe for unknown signal frequency.

The phase processors have CFAR capability; i.e., a threshold can be fixed to realize a prescribed P_F for a given value of M, regardless of the absolute noise level. The threshold for many of the processors can be determined by means of the exact formulas in appendix C.

Approximations to system detection capability derived in appendix C can be fruitfully employed to evaluate processor performance for other values of M and SNR. However, for small values of M or very small values of P_F, the approxi-

mations are less accurate. A guide as to when approximations are valid is offered by the DCs presented in this report. When simulation becomes expensive, in terms of time and storage for larger M, analytical results are better and are the recommended approach to system evaluation.

The investigation here has assumed that only one sine wave is present. If a second sine wave, separated in frequency by more than the width of an NBF in the preprocessor, is present, the desired tone will dominate the NBF output and the current results will apply. Similarly, if the preprocessor consists of a sequence of FFTs, separation in frequency of the sine waves by more than the inverse of the time duration used in an individual FFT will not cause problems. However, if the two tones lie within the same NBF width and are of comparable strength, phase-processing can suffer severely because measurements of phase are affected by both sine waves and will vary greatly with time. In this situation, either narrower filters should be used or amplitude processing, such as Processor III, should be used. This situation should be investigated more fully, probably via simulation.

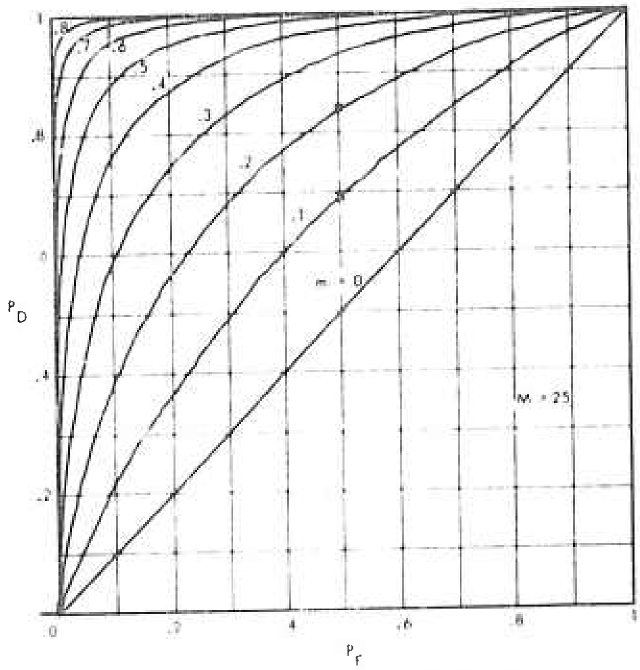

Figure 1A. Receiver Operating Characteristic for $P_F \leq 1$

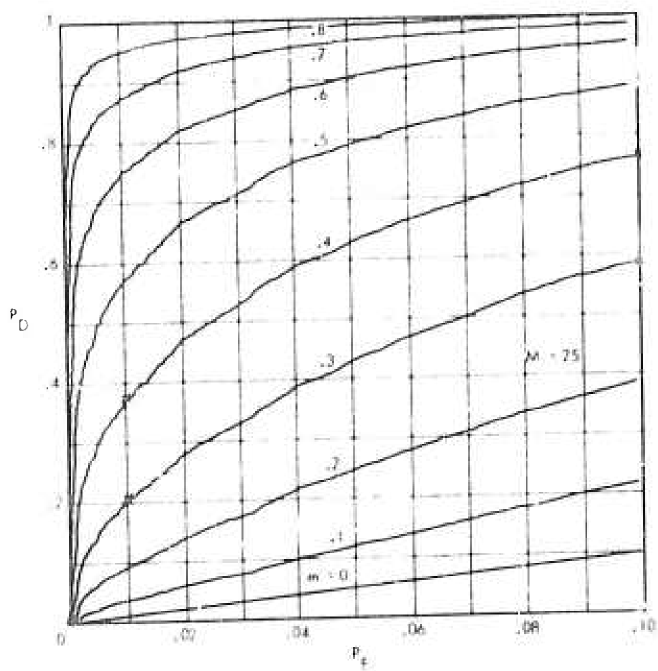

Figure 1B. Receiver Operating Characteristic for $P_F \leq .1$

Figure 1. Receiver Operating Characteristic for Processor I, Known Signal Frequency

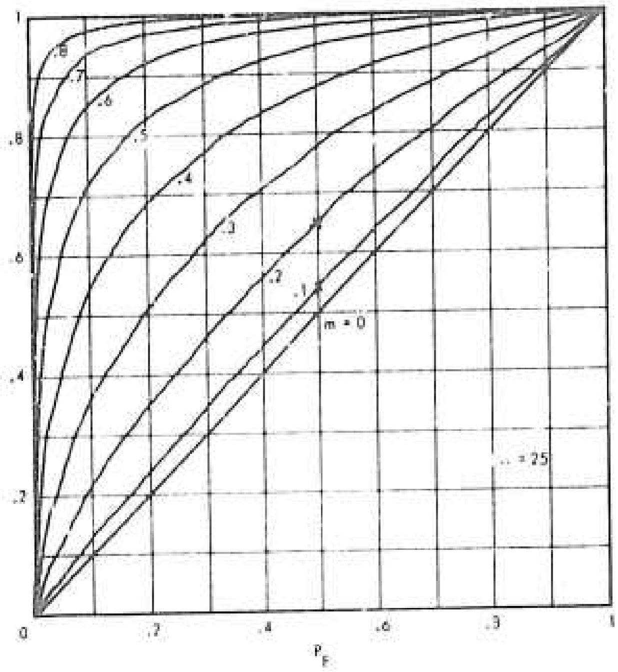

Figure 2A. Receiver Operating Characteristic for $P_F \leq 1$

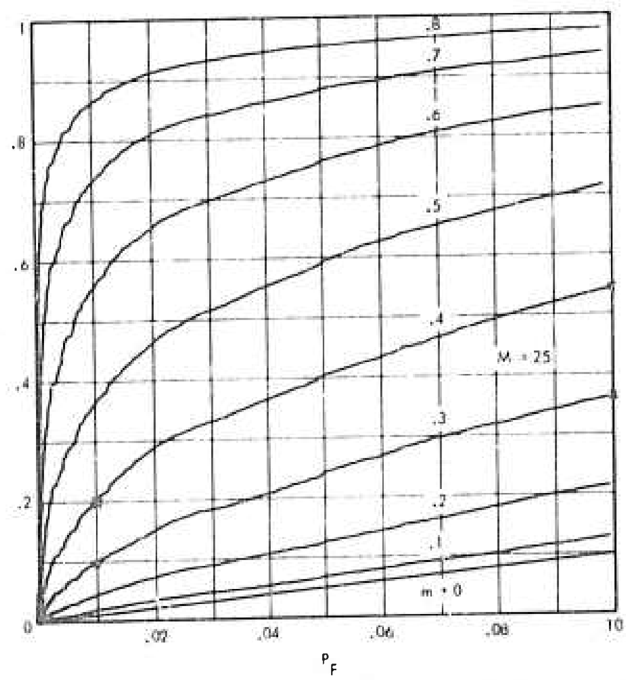

Figure 2B. Receiver Operating Characteristic for $P_F \leq .1$

Figure 2. Receiver Operating Characteristic for Processor II, Known Signal Frequency

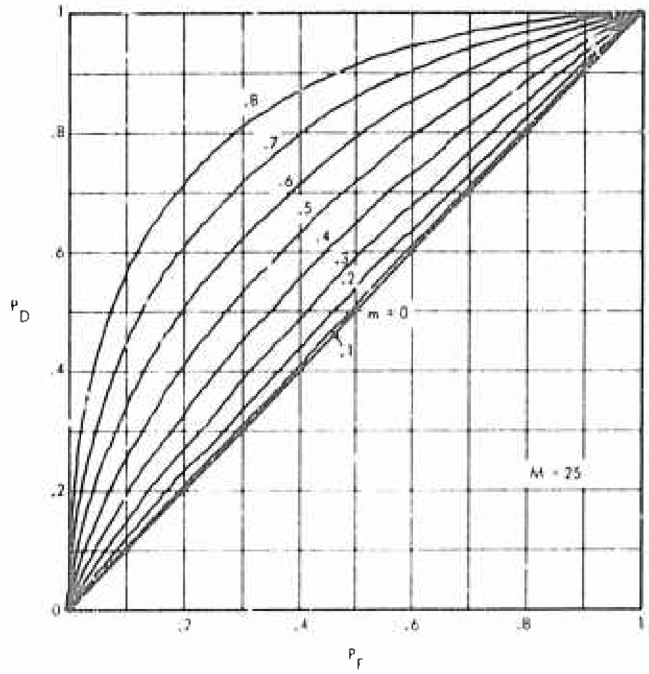

Figure 3A. Receiver Operating Characteristic for $P_F \leq 1$

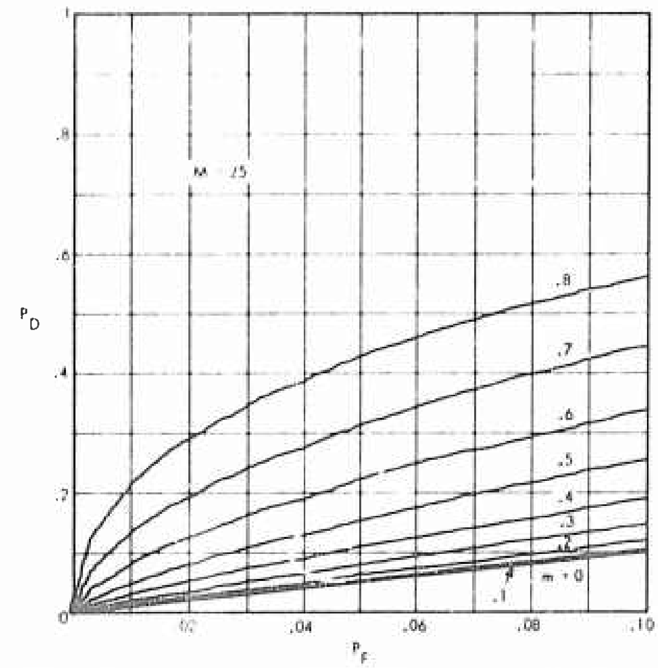

Figure 3B. Receiver Operating Characteristic for $P_F \leq .1$

Figure 3. Receiver Operating Characteristic for Processor III, Known Signal Frequency

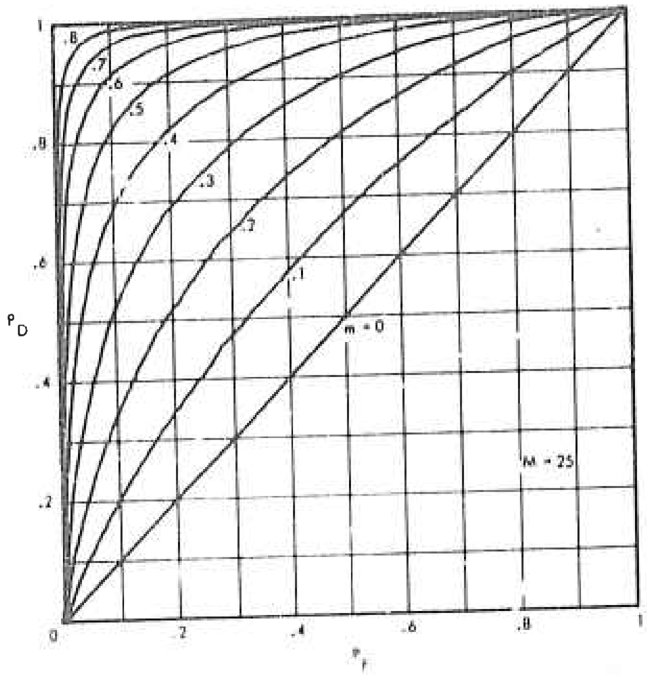

Figure 4A. Receiver Operating Characteristic for $P_F \leq 1$

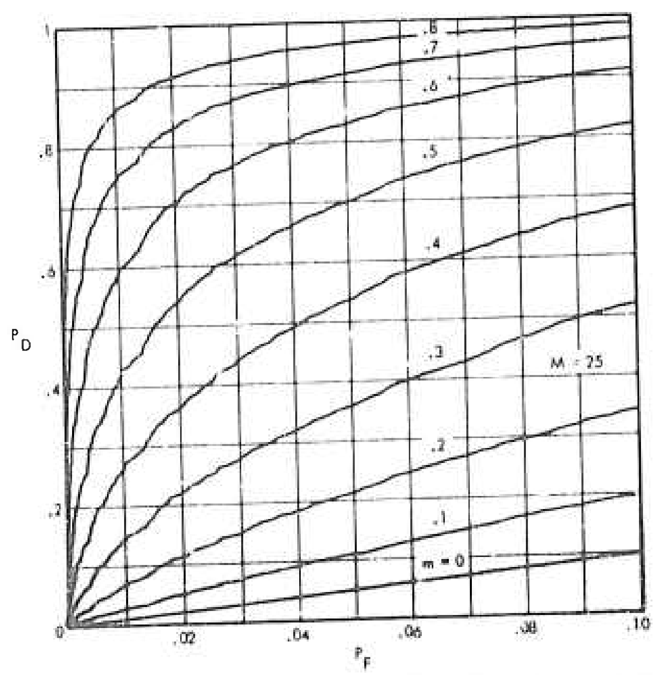

Figure 4B. Receiver Operating Characteristic for $P_F \leq .1$

Figure 4. Receiver Operating Characteristic for Processor IV, Known Signal Frequency

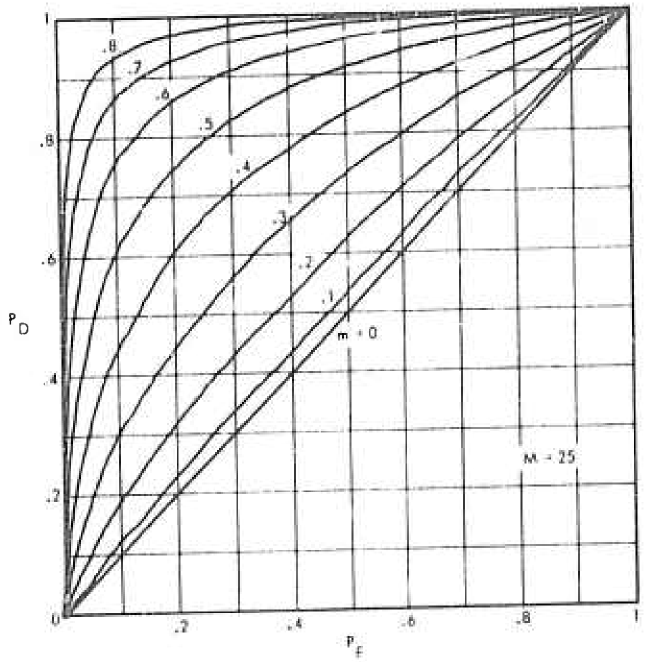

Figure 5A. Receiver Operating Characteristic for $P_F \leq 1$

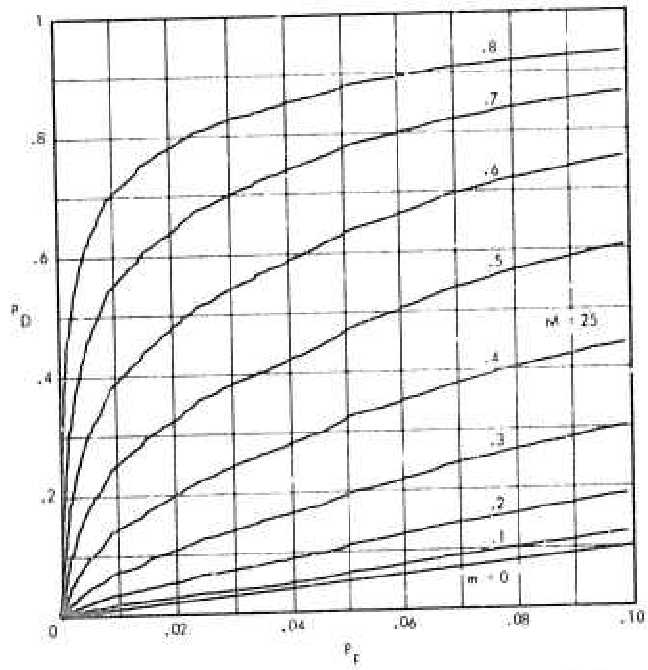

Figure 5B. Receiver Operating Characteristic for $P_F \leq .1$

Figure 5. Receiver Operating Characteristic for Processor V, Known Signal Frequency

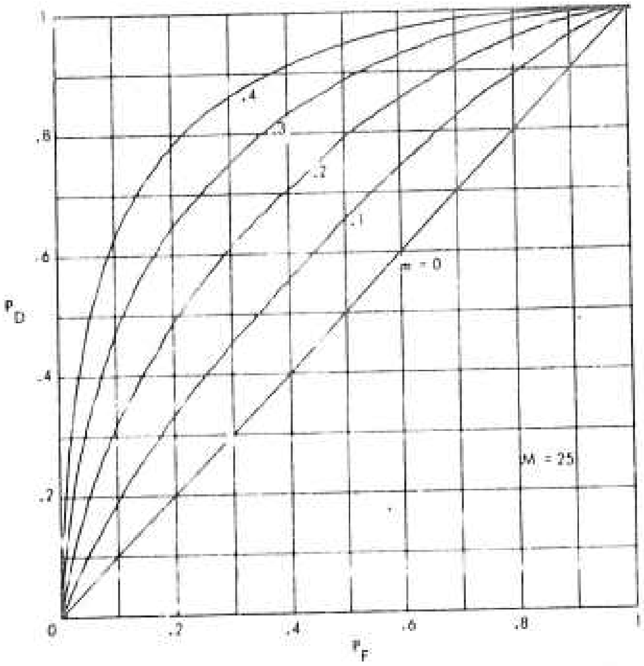

Figure 6A. Receiver Operating Characteristic
for $\psi_0 = 0$; Band $(-\pi, \pi)$

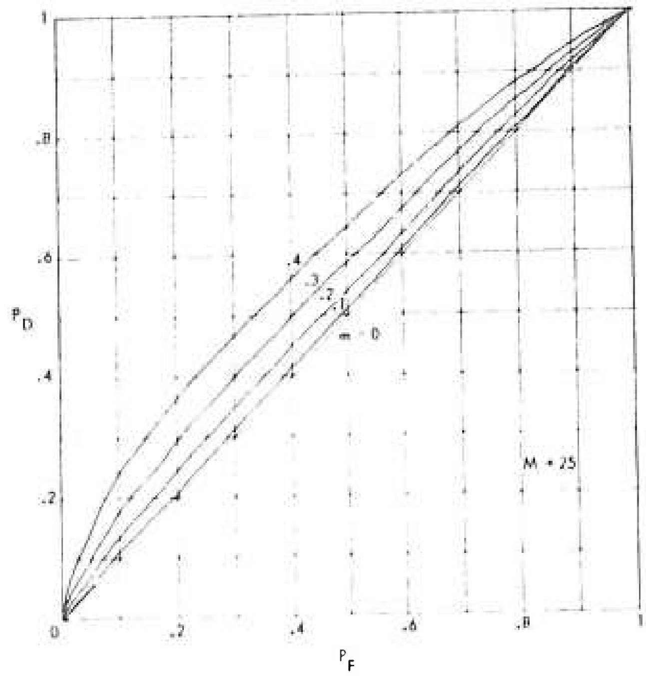

Figure 6B. Receiver Operating Characteristic
for $\psi_0 = \pi/2$; Band $(-\pi, \pi)$

Figure 6. Receiver Operating Characteristic for Processor VI, Known Signal Frequency

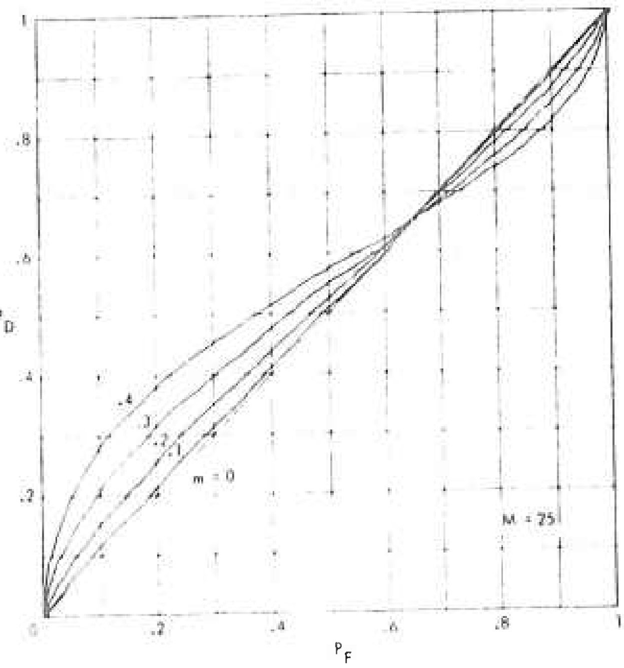

Figure 6C. Receiver Operating Characteristic for $\psi_0 \in (0, 2\pi)$; Band $(-\pi, \pi)$

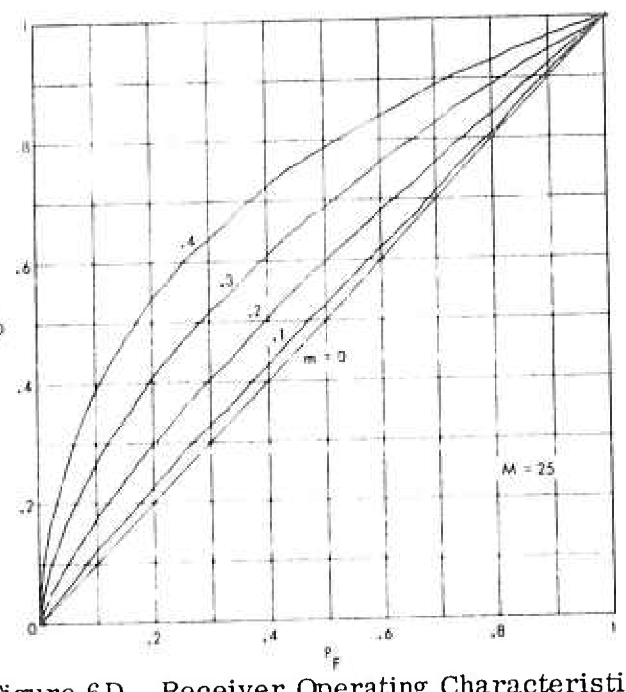

Figure 6D. Receiver Operating Characteristic for $\psi_0 \in (0, 2\pi)$; Bands $(-\pi, \pi)$ and $(0, 2\pi)$

Figure 6 (Cont'd). Receiver Operating Characteristic for Processor VI, Known Signal Frequency

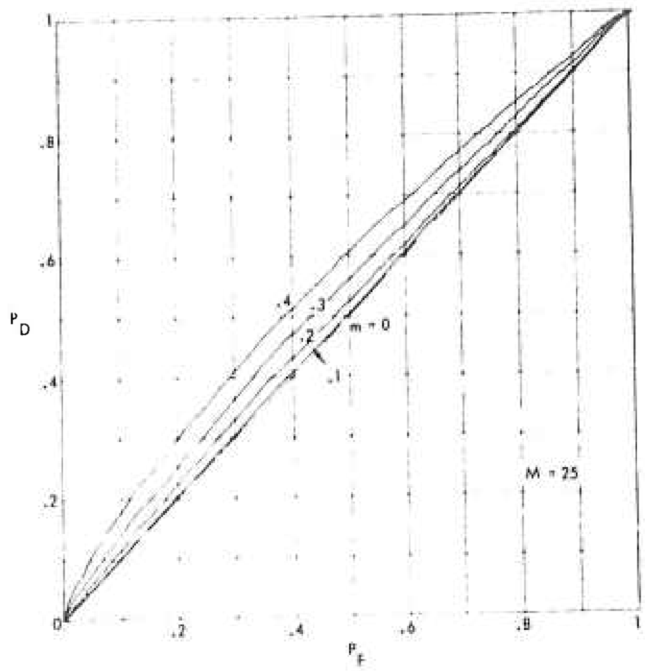

Figure 6E. Receiver Operating Characteristic for $\psi_0 \epsilon (0, 2\pi)$; Best Tilted Line

Figure 6 (Cont'd). Receiver Operating Characteristic for Processor VI, Known Signal Frequency

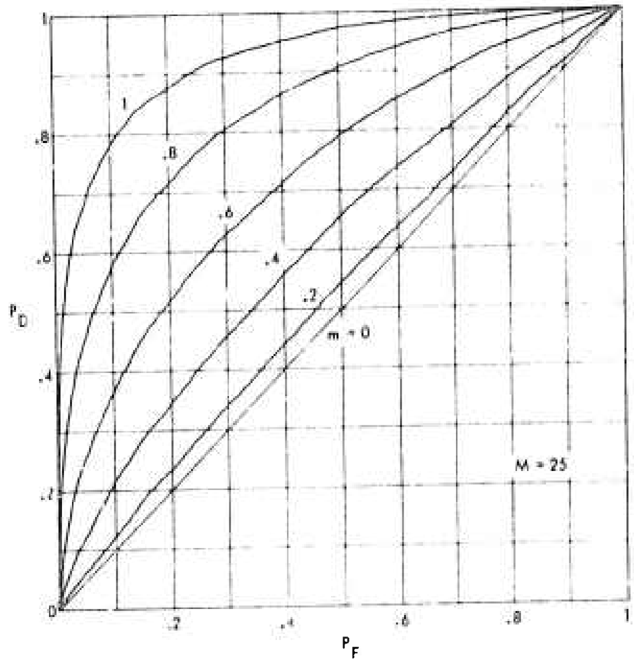

Figure 7A. $\mu = 0$

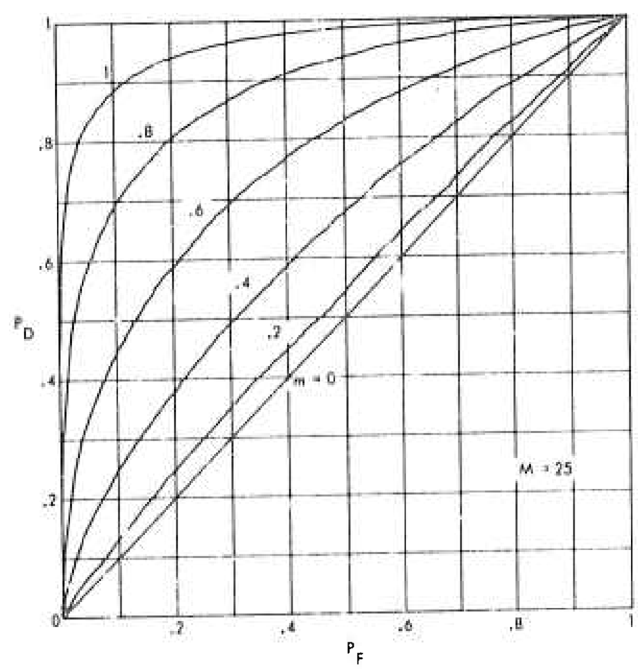

Figure 7B. $\mu = .5$

Figure 7. Receiver Operating Characteristic for Processor VII, Known Signal Frequency

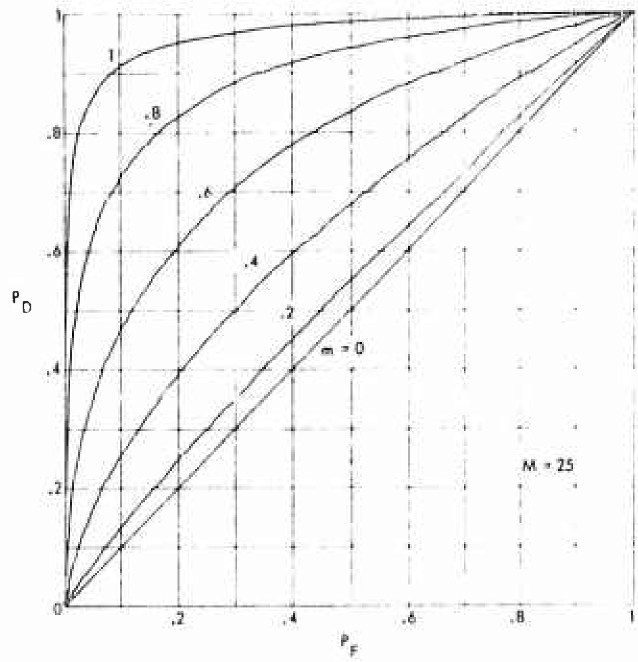

Figure 7C. $\mu = 1$

Figure 7 (Cont'd). Receiver Operating Characteristic for Processor VII, Known Signal Frequency

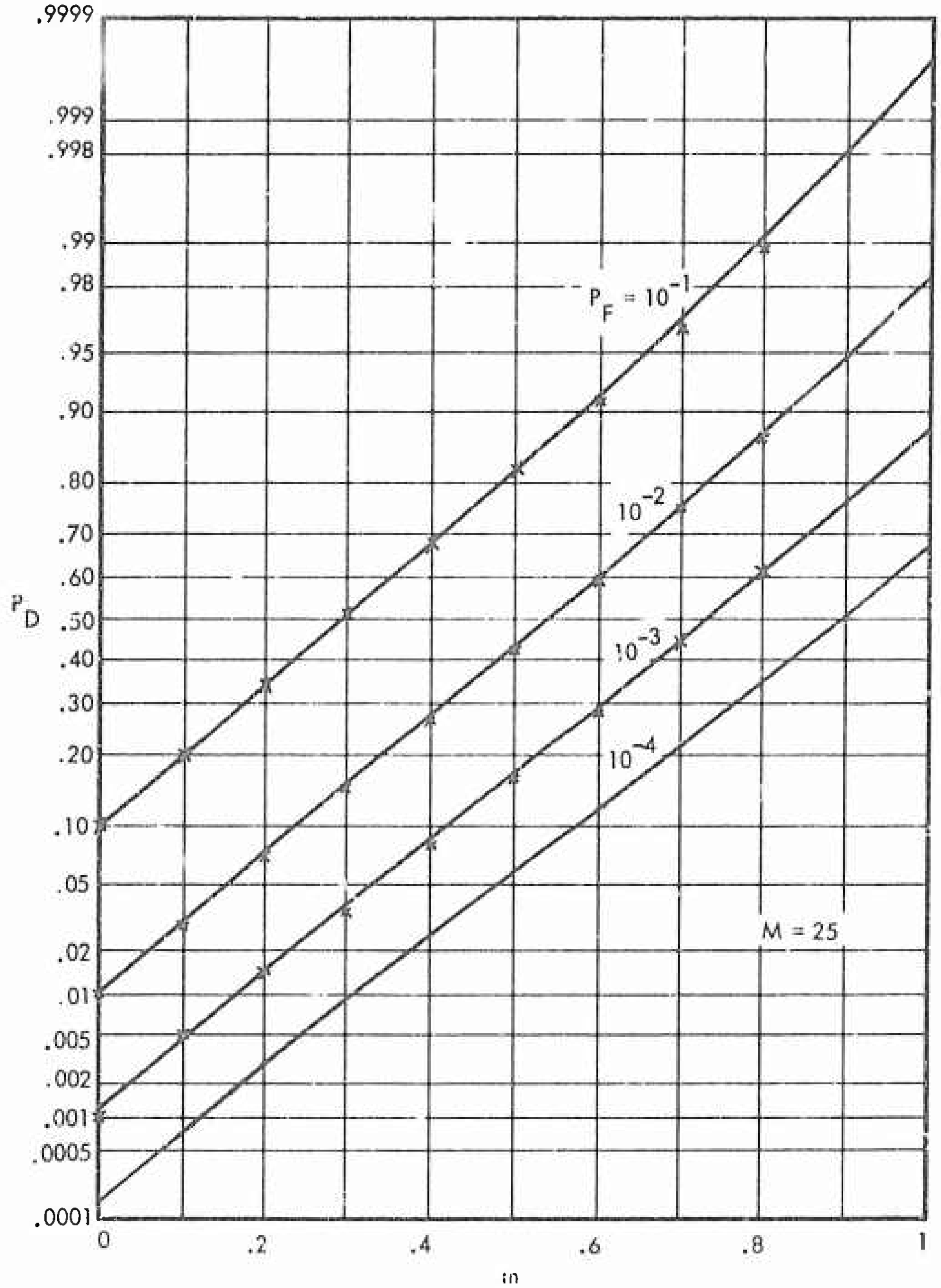

Figure 8. Detection Characteristic for Processor IV, Known Signal Frequency

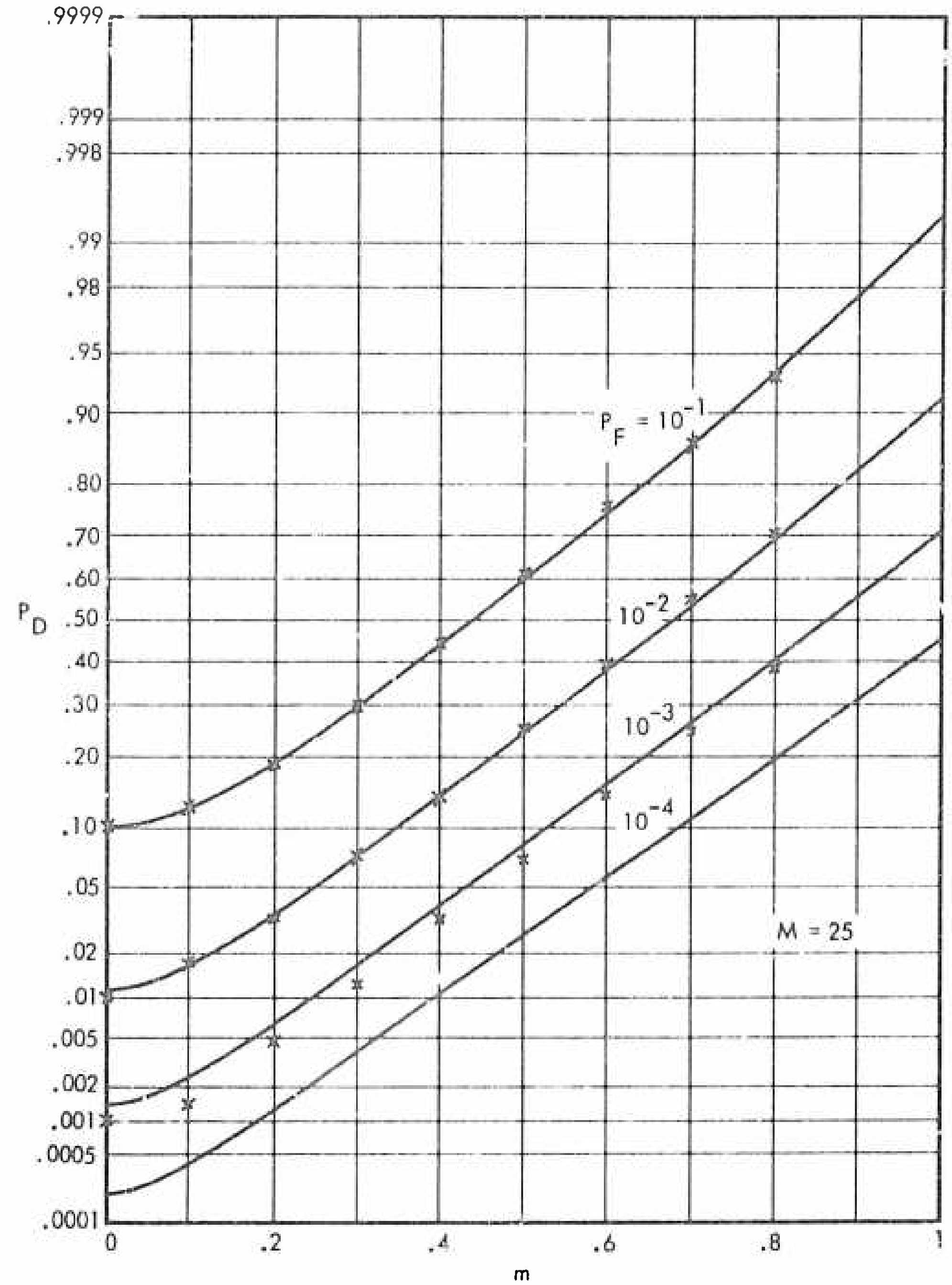

Figure 9. Detection Characteristic for Processor V, Known Signal Frequency

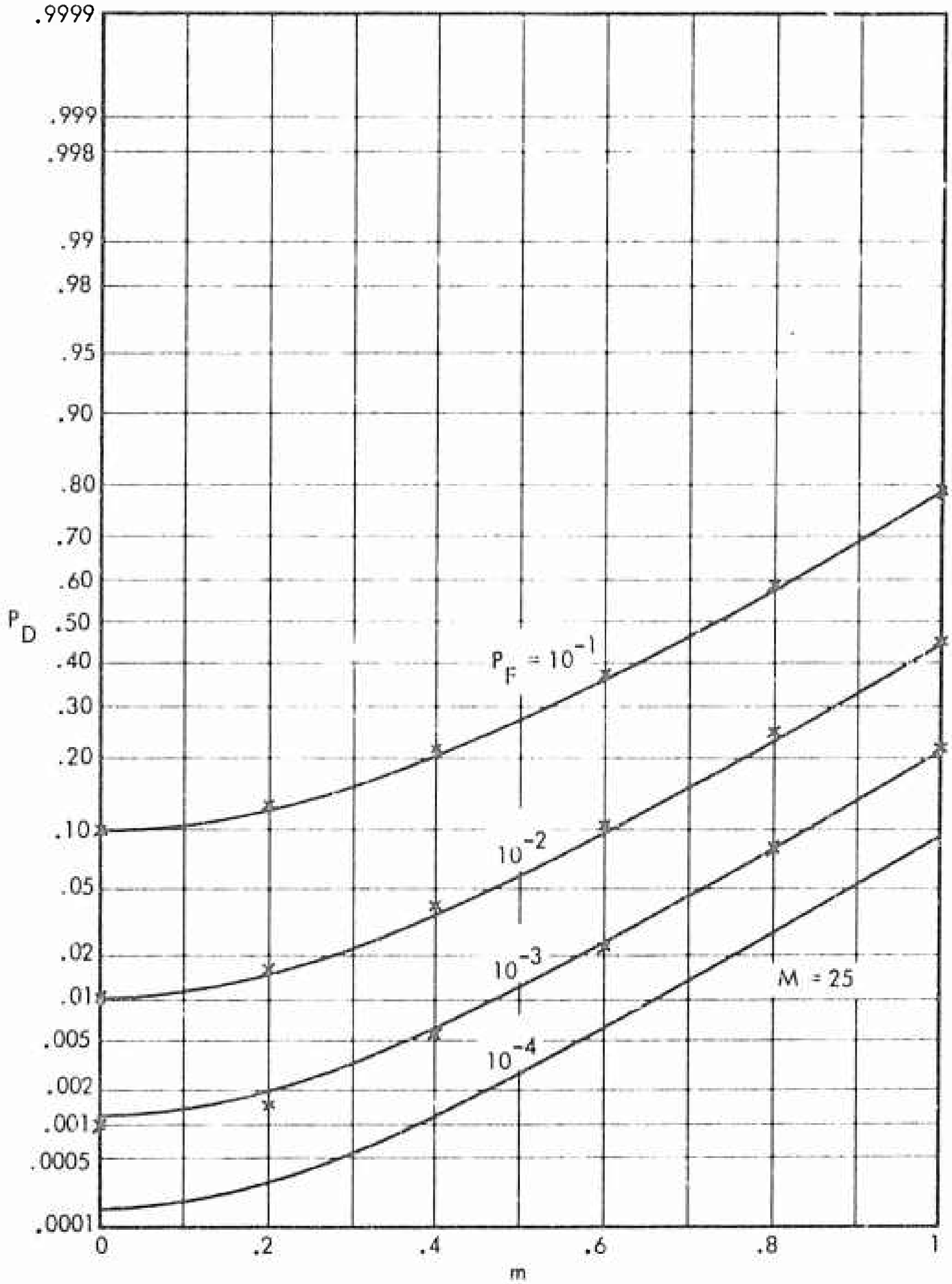

Figure 10A. $\mu = 0$

Figure 10. Detection Characteristic for Processor VII, Known Signal Frequency

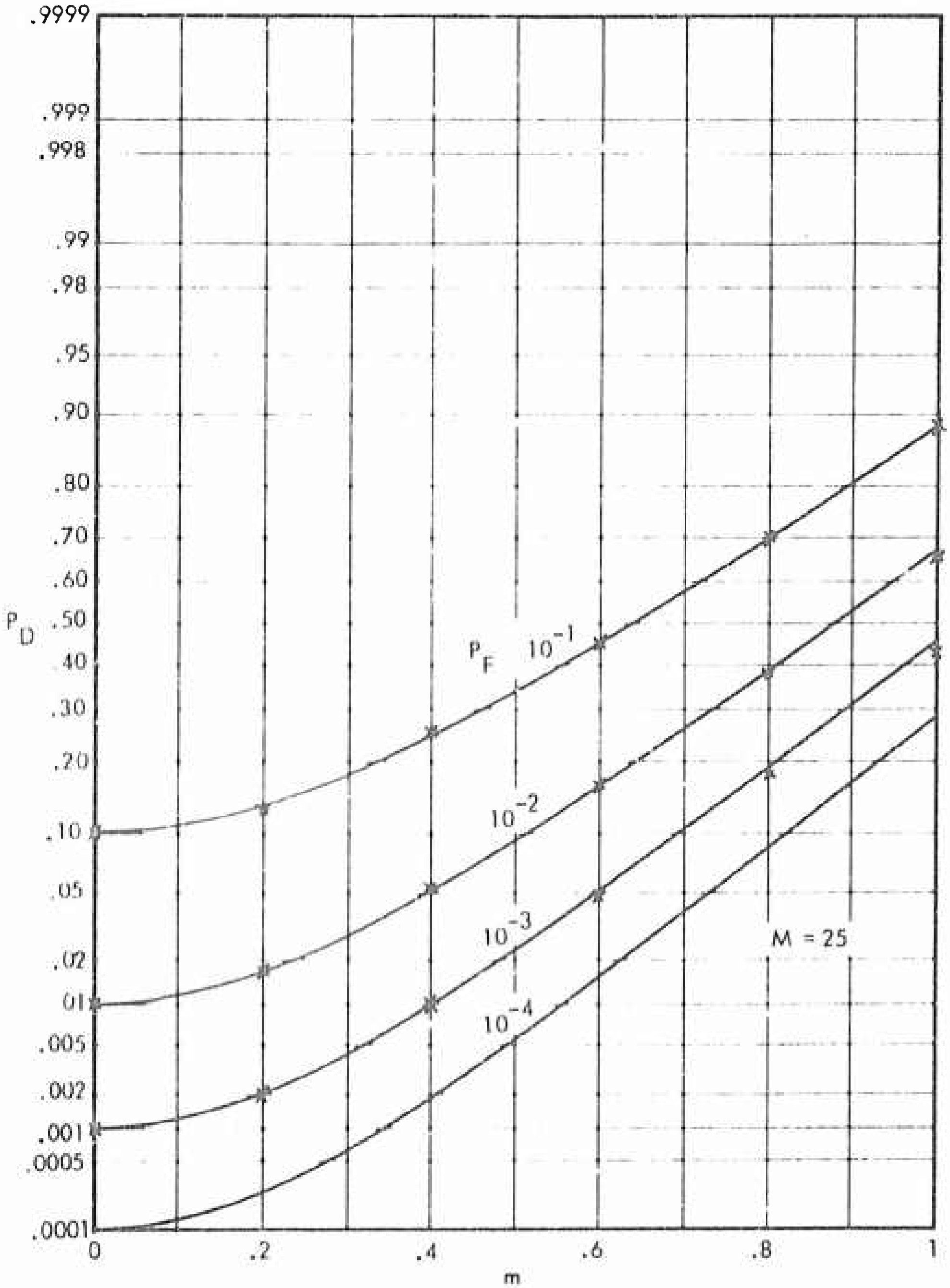

Figure 10B. $\mu = .5$

Figure 10 (Cont'd). Detection Characteristic for Processor VII, Known Signal Frequency

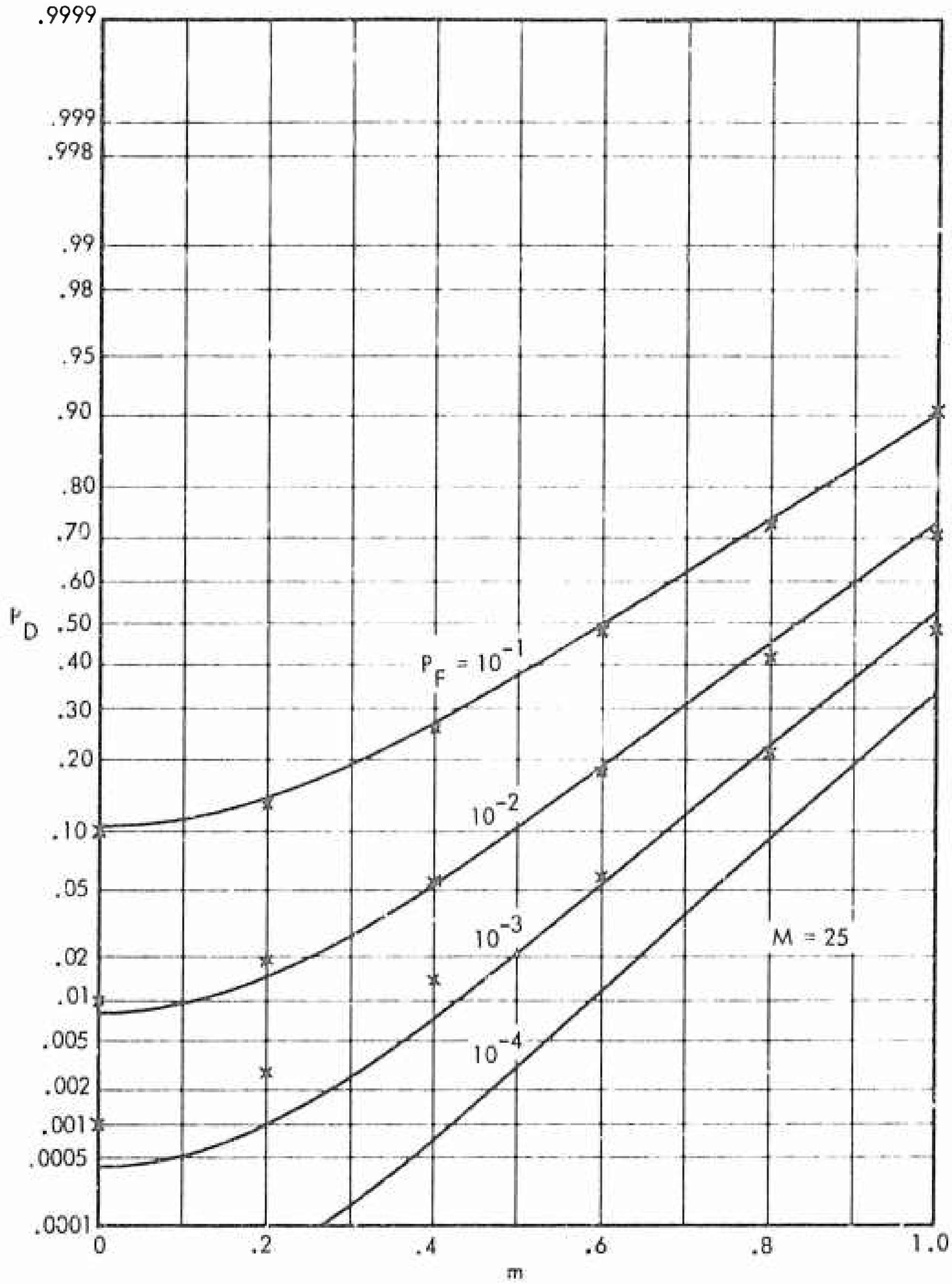

Figure 10C. $\mu = 1$

Figure 10 (Cont'd). Detection Characteristic for Processor VII, Known Signal Frequency

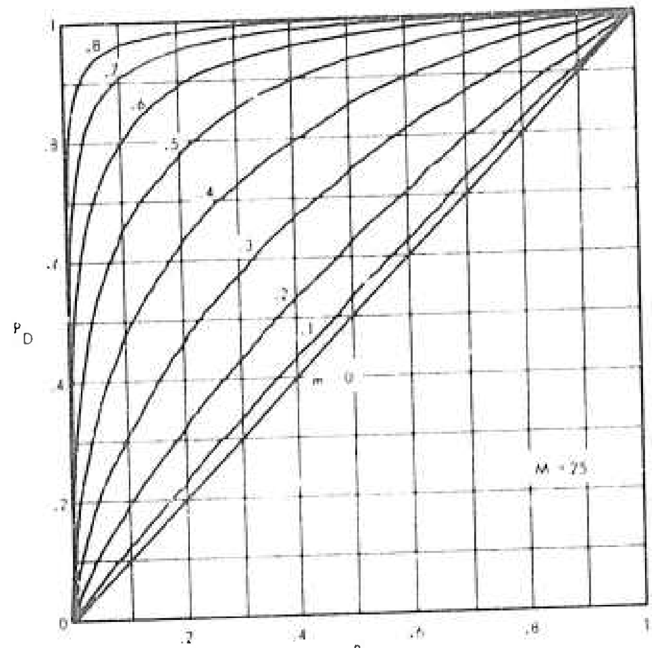

Figure 11A. Receiver Operating Characteristic for $P_F \leq 1$

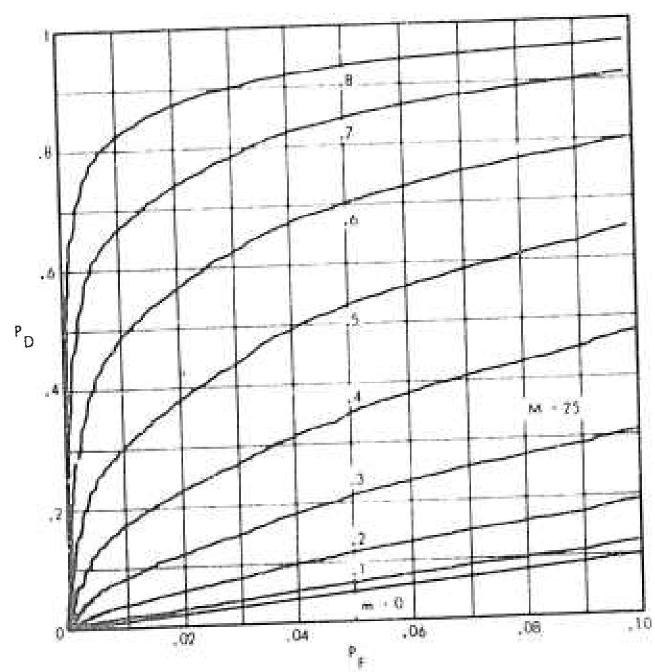

Figure 11B. Receiver Operating Characteristic for $P_F \leq .1$

Figure 11. Receiver Operating Characteristic for Processor II, Unknown Signal Frequency; Track

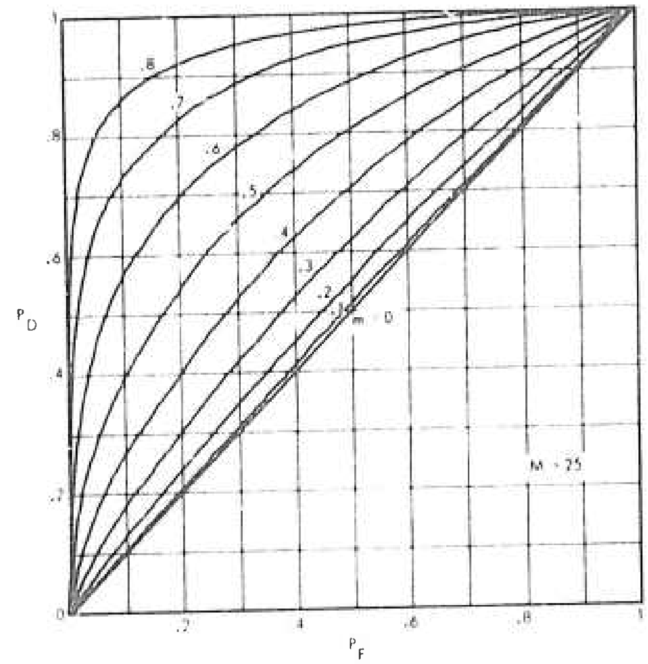

Figure 12A. Receiver Operating Characteristic for $P_F \leq 1$

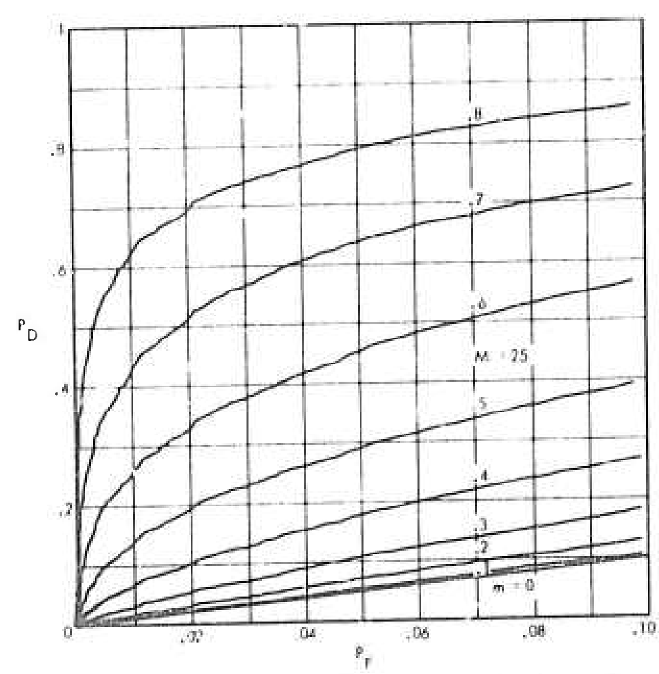

Figure 12B. Receiver Operating Characteristic for $P_F \leq .1$

Figure 12. Receiver Operating Characteristic for Processor II, Unknown Signal Frequency; Search

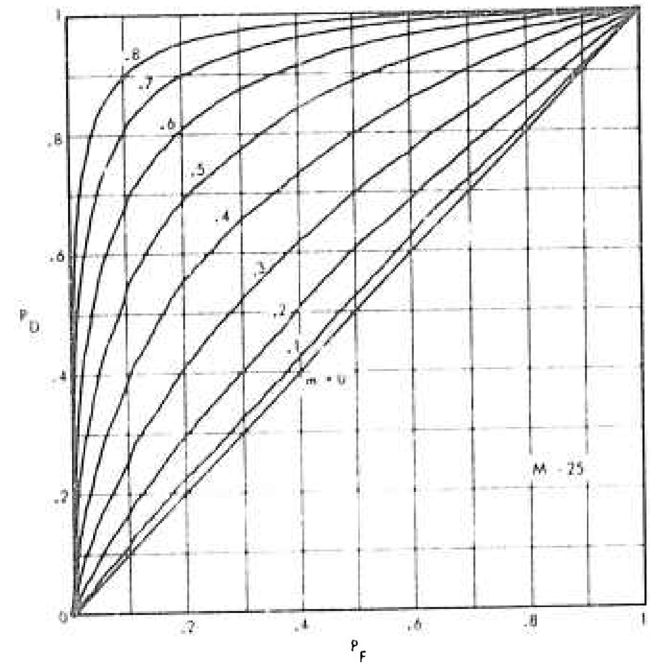

Figure 13A. Receiver Operating Characteristic for $P_F \leq 1$

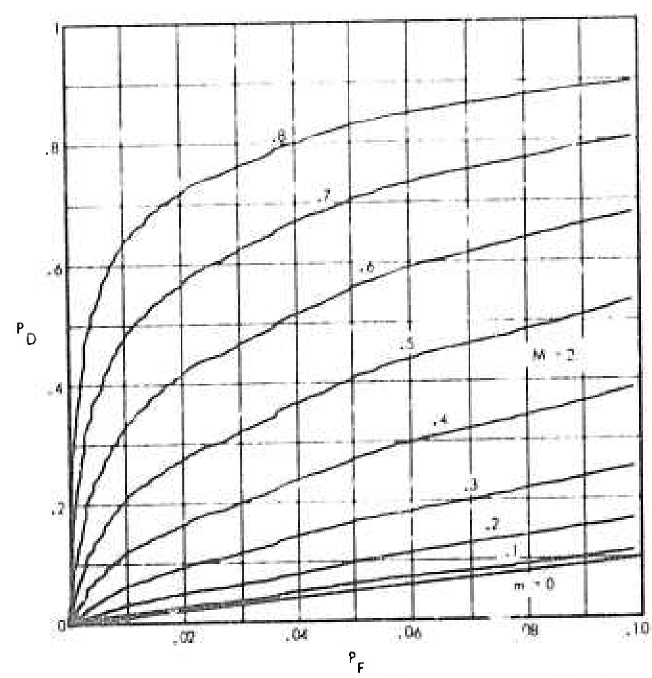

Figure 13B. Receiver Operating Characteristic for $P_F \leq .1$

Figure 13. Receiver Operating Char .cteristic for Processor V, Unknown Signal Frequency; Track

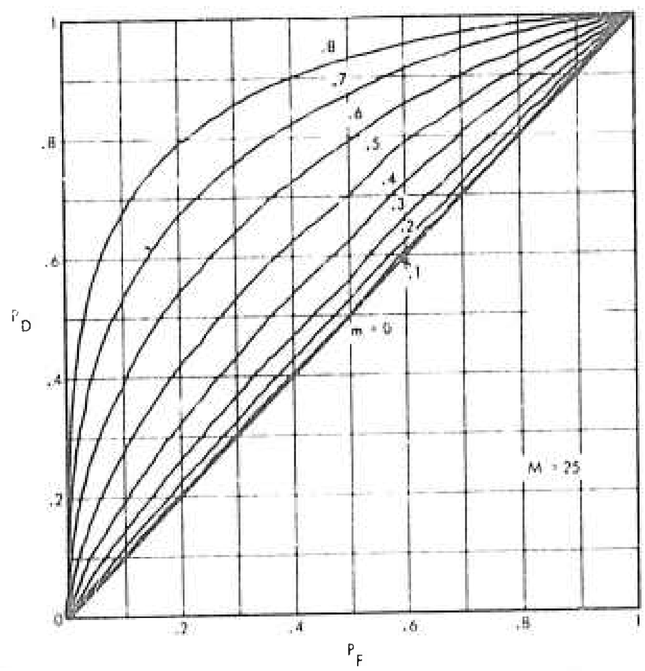

Figure 14A. Receiver Operating Characteristic for $P_F \leq 1$

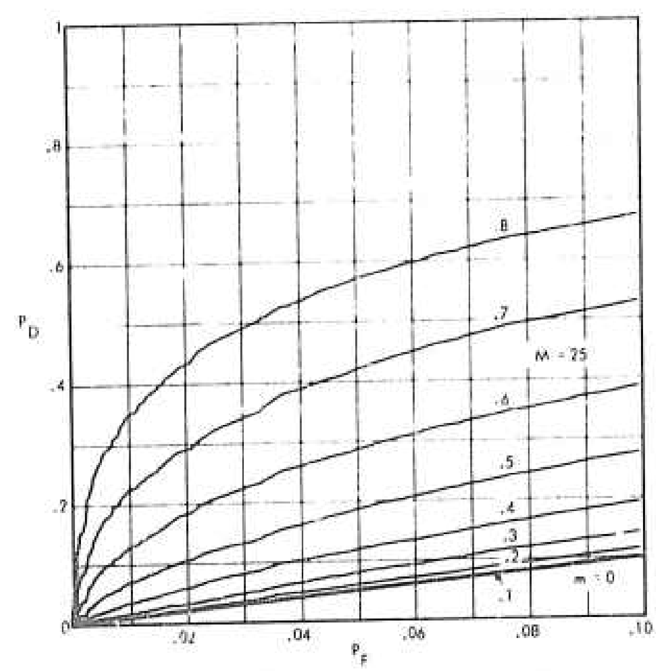

Figure 14B. Receiver Operating Characteristic for $P_F \leq .1$

Figure 14. Receiver Operating Characteristic for Processor V, Unknown Signal Frequency; Search

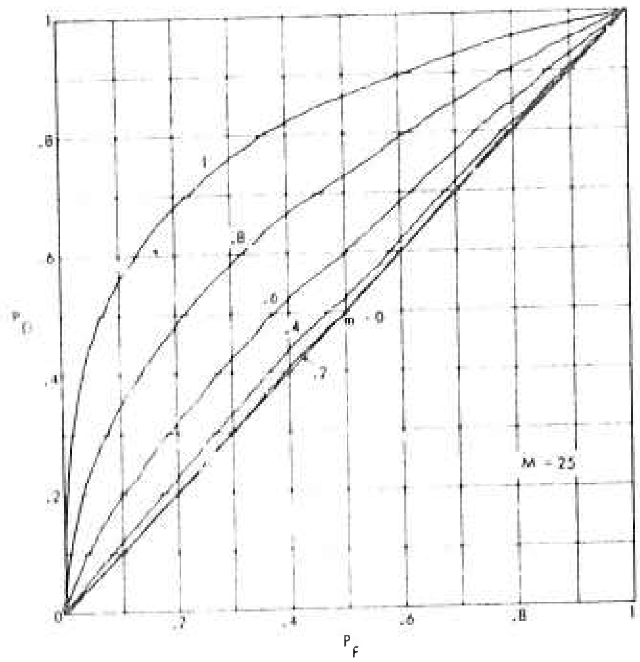

Figure 15A. $\mu = 0$

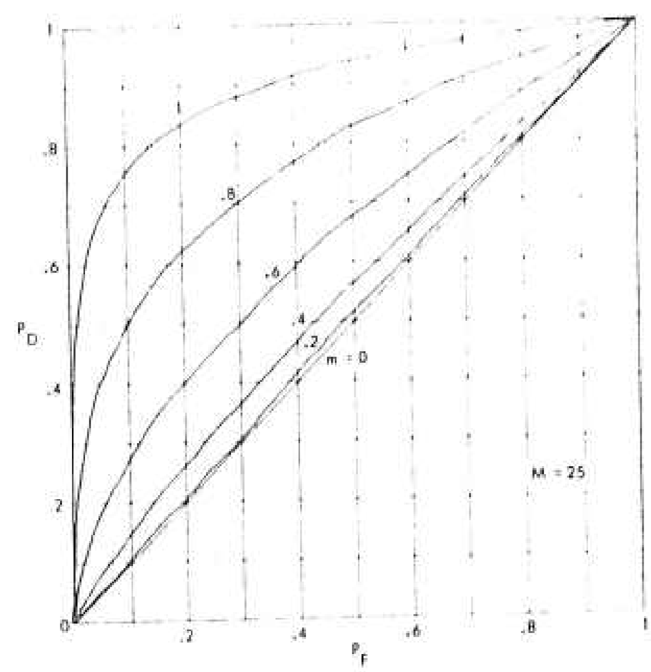

Figure 15B. $\mu = .5$

Figure 15. Receiver Operating Characteristic for Processor VII, Unknown Signal Frequency

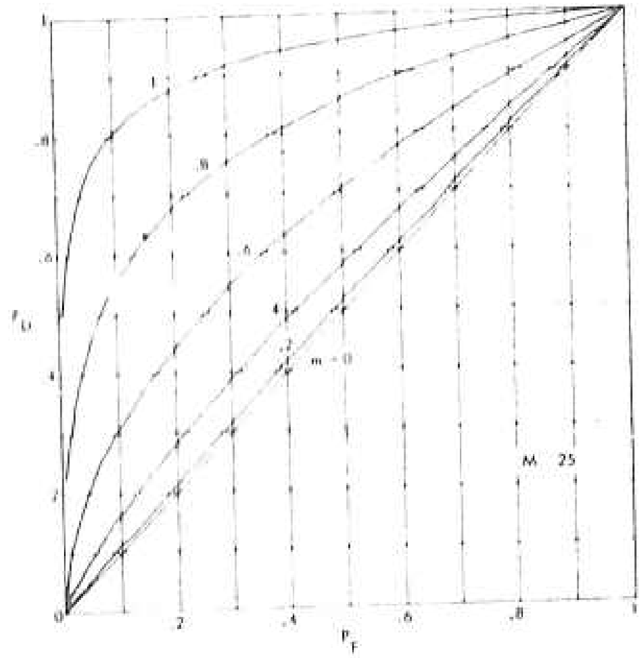

Figure 15C. $\mu = 1$

Figure 15 (Cont'd). Receiver Operating Characteristic for Processor VII, Unknown Signal Frequency

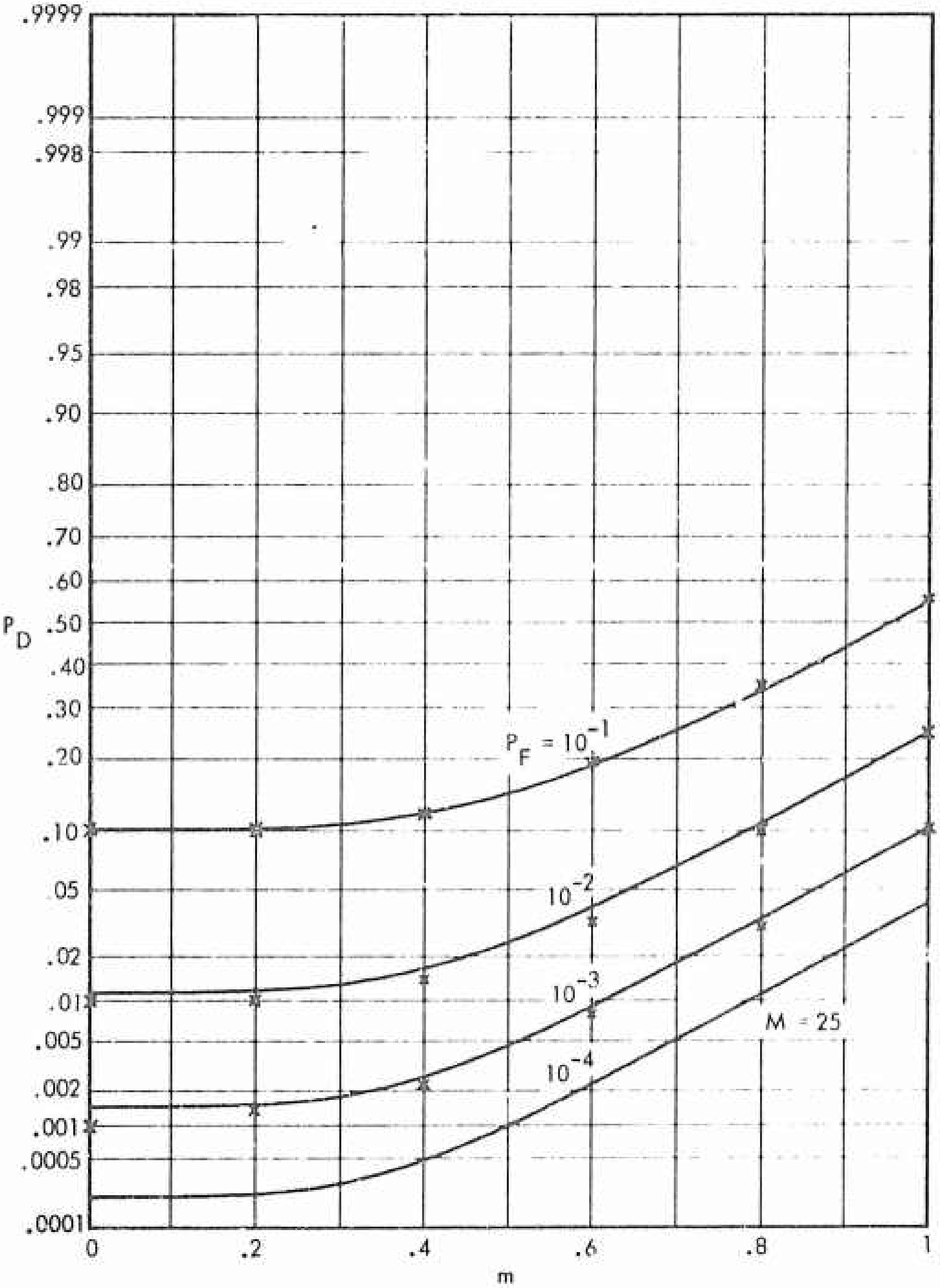

Figure 16A. $\mu = 0$

Figure 16. Detection Characteristic for Processor VII, Unknown Signal Frequency

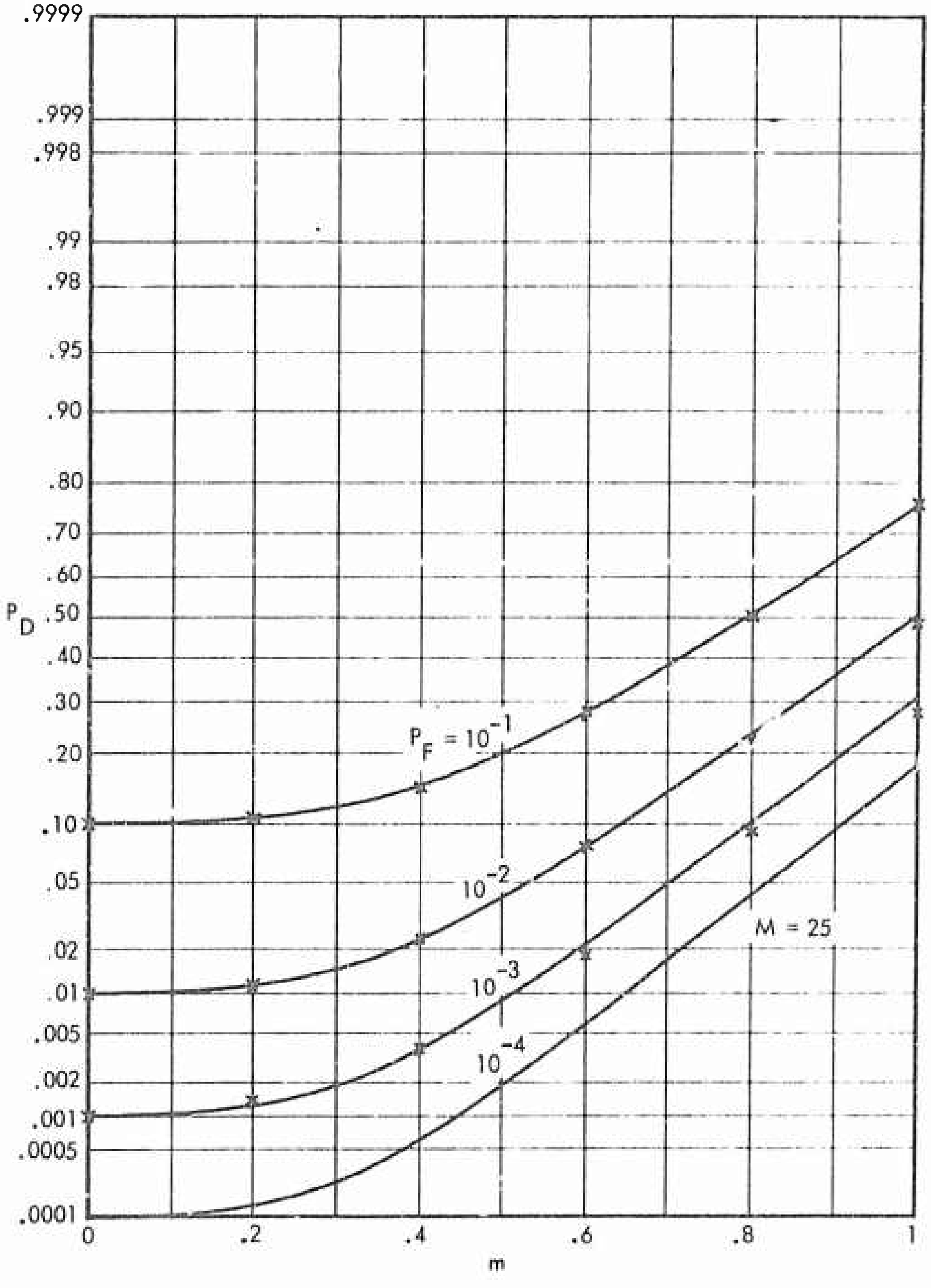

Figure 16B. $\mu = .5$

Figure 16 (Cont'd). Detection Characteristic for Processor VII, Unknown Signal Frequency

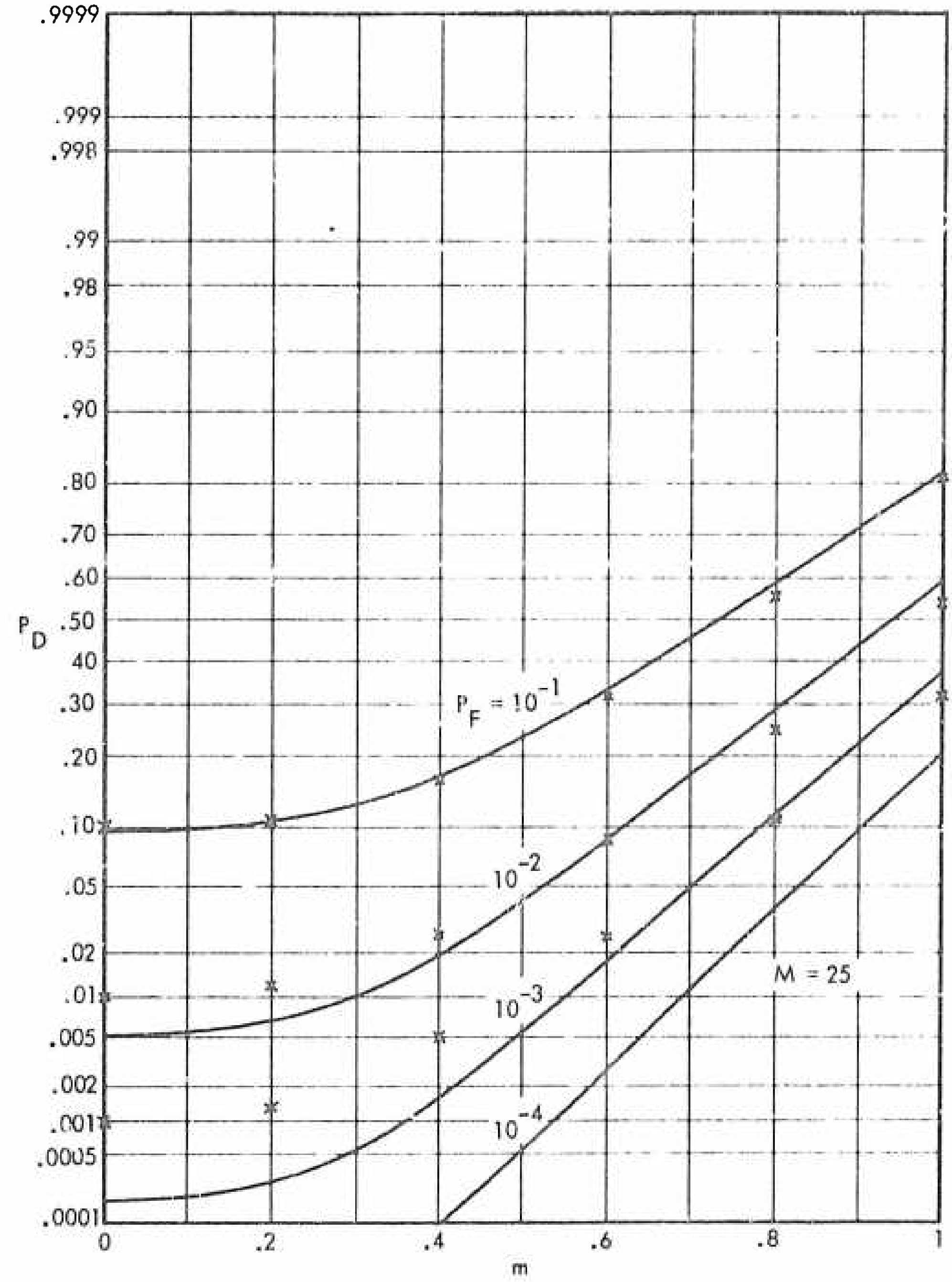

Figure 16C. $\mu = 1$

Figure 16 (Cont'd). Detection Characteristic for Processor VII, Unknown Signal Frequency

Appendix A

DERIVATIONS OF PROCESSORS

The processors will be discussed in the same order in the appendixes as they were in the text. Also, the appendixes are arranged such that particular processors have the same section number in each; e.g., Processor V for known signal frequency is treated in sections A.1.5, B.1.5, and C.1.5.

A.1 KNOWN SIGNAL FREQUENCY

A.1.1 PROCESSOR I: KNOWN SIGNAL PHASE; AMPLITUDE AND PHASE SAMPLES

The PDF of random variables

$$\underset{\sim}{x} \equiv \left[x_1 \cdots x_M\right]^T, \quad \underset{\sim}{y} \equiv \left[y_1 \cdots y_M\right]^T \tag{A-1}$$

for signal present is available from (3), (4), and (5) as

$$p_1(\underset{\sim}{x}, \underset{\sim}{y}) = \prod_{k=1}^{M} \left\{ \frac{1}{2\pi\sigma^2} \exp\left[- \frac{(x_k - P_o \cos\psi_o)^2 + (y_k - P_o \sin\psi_o)^2}{2\sigma^2} \right] \right\} . \tag{A-2}$$

The PDF for signal absent, $p_0(\underset{\sim}{x}, \underset{\sim}{y})$, is obtained from (A-2) by setting $P_r = 0$. The likelihood ratio (LR) is the ratio of p_1 to p_0 and can, using (5), be put in the form

$$LR = \exp\left[\frac{P_o}{\sigma^2} \operatorname{Re}\left\{ e^{-i\psi_o} \sum_{k=1}^{M} R_k e^{i\theta_k} \right\} - \frac{M P_o^2}{2\sigma^2} \right] . \tag{A-3}$$

Therefore, the LR test is*

$$\operatorname{Re}\left\{ e^{-i\psi_o} \sum_{k=1}^{M} R_k e^{i\theta_k} \right\} \gtrless T , \tag{A-4}$$

where threshold T can be chosen for a specified false alarm probability. Notice that even though P_o is assumed known in (A-2), it is not used in the LR test (A-4).

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*Satisfaction of the upper inequality leads to the decision that a signal is present; satisfaction of the lower inequality yields the decision that the signal is absent.

A.1.2 PROCESSOR II: UNKNOWN SIGNAL PHASE; AMPLITUDE AND PHASE SAMPLES

When the signal phase is unknown, the PDF of $\underset{\sim}{x}$ and $\underset{\sim}{y}$ for a hypothesized signal phase ψ is

$$p_1(\underset{\sim}{x}, \underset{\sim}{y} | \psi) = \prod_{k=1}^{M} \left\{ \frac{1}{2\pi\sigma^2} \exp\left[- \frac{(x_k - P_o \cos\psi)^2 + (y_k - P_o \sin\psi)^2}{2\sigma^2} \right] \right\}. \tag{A-5}$$

If the a priori PDF of ψ is uniform, there follows

$$p_1(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{1}{2\pi} \int_{2\pi} d\psi \; p_1(\underset{\sim}{x}, \underset{\sim}{y} | \psi)$$

$$= \frac{1}{(2\pi\sigma^2)^M} \exp\left[- \frac{\sum_{k=1}^{M} R_k^2 + M P_o^2}{2\sigma^2} \right] I_o\left(\frac{P_o}{\sigma^2} \left| \sum_{k=1}^{M} R_k e^{i\theta_k} \right| \right). \tag{A-6}$$

The PDF for signal absent is obtained by setting $P_o = 0$ in (A-6). The average LR is then given by

$$\exp\left(- \frac{M P_o^2}{2\sigma^2} \right) I_o\left(\frac{P_o}{\sigma^2} \left| \sum_{k=1}^{M} R_k e^{i\theta_k} \right| \right). \tag{A-7}$$

Therefore, the LR test is

$$\left| \sum_{k=1}^{M} R_k e^{i\theta_k} \right| \gtrless T. \tag{A-8}$$

Again, although P_o is assumed known in (A-5), it is not used in the LR test.

Instead of assuming a uniform PDF for hypothesized phase ψ, it is possible to choose ψ as the ML estimate in (A-5). By rearranging (A-5) in the form

$$p_1(\underset{\sim}{x}, \underset{\sim}{y} | \psi) = \frac{1}{(2\pi\sigma^2)^M} \exp\left[- \frac{\sum_{k=1}^{M} R_k^2 + M P_o^2}{2\sigma^2} + \frac{P_o}{\sigma^2} \mathrm{Re}\left\{ e^{-i\psi} \sum_{k=1}^{M} R_k e^{i\theta_k} \right\} \right], \tag{A-9}$$

it is readily apparent that p_1 is maximized by the ML estimate

$$\hat{\psi} = \arg\left\{ \sum_{k=1}^{M} R_k e^{i\theta_k} \right\}. \tag{A-10}$$

The maximum value of p_1 is then given by

$$p_1(\underset{\sim}{x}, \underset{\sim}{y} | \hat{\psi}) = \frac{1}{(2\pi\sigma^2)^M} \exp\left[- \frac{\sum_{k=1}^{M} R_k^2 + M P_o^2}{2\sigma^2} + \frac{P_o}{\sigma^2} \left| \sum_{k=1}^{M} R_k e^{i\theta_k} \right| \right]. \tag{A-11}$$

The generalized LR is the ratio of (A-11) to p_0:

$$\exp\left[-\frac{MP_0^2}{2\sigma^2}+\frac{P_0}{\sigma^2}\left|\sum_{K=1}^{M}R_K e^{i\theta_K}\right|\right]. \tag{A-12}$$

The generalized LR test is therefore identical to (A-8). Thus two different methods of treating signal phase yield the same processor (A-8).

A.1.3 PROCESSOR III: INDEPENDENT SIGNAL PHASES; AMPLITUDE AND PHASE SAMPLES

When all signal phases are independent from sample to sample, the PDF of $\underset{\sim}{x}$ and $\underset{\sim}{y}$ is given by a modified form of (A-2), namely,

$$p_1(\underset{\sim}{x},\underset{\sim}{y}|\underset{\sim}{\psi})=\prod_{K=1}^{M}\left\{\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(x_K-P_0\cos\psi_K)^2+(y_K-P_0\sin\psi_K)^2}{2\sigma^2}\right]\right\}, \tag{A-13}$$

where $\underset{\sim}{\psi}=[\psi_1\cdots\psi_M]^T$ is the set of M random signal phases. For uniform PDFs of each ψ_K, there follows

$$\begin{aligned} p_1(\underset{\sim}{x},\underset{\sim}{y}) &= \frac{1}{(2\pi)^M}\int_{2\pi}\cdots\int d\psi_1\cdots d\psi_M\, p_1(\underset{\sim}{x},\underset{\sim}{y}|\underset{\sim}{\psi}) \\ &= \frac{1}{(2\pi\sigma^2)^M}\exp\left[-\frac{\sum_{K=1}^{M}R_K^2+MP_0^2}{2\sigma^2}\right]\prod_{K=1}^{M}I_0\left(\frac{P_0}{\sigma^2}R_K\right). \end{aligned} \tag{A-14}$$

The average LR follows readily from (A-14) as

$$\exp\left(-\frac{MP_0^2}{2\sigma^2}\right)\prod_{K=1}^{M}I_0\left(\frac{P_0}{\sigma^2}R_K\right), \tag{A-15}$$

and the LR test becomes

$$\sum_{K=1}^{M}\ln I_0\left(\frac{P_0}{\sigma^2}R_K\right)\gtrless T. \tag{A-16}$$

This test makes no use of phase samples θ_K, but does depend on knowledge of P_0; however, for small SNR (small P_0/σ), (A-16) becomes

$$\sum_{K=1}^{M}R_K^2\gtrless T. \tag{A-17}$$

For small SNR, this approximate LR test does not require knowledge of P_0.

A.1.4 PROCESSOR IV: KNOWN SIGNAL PHASE; PHASE SAMPLES

The PDF of amplitude and phase samples R_k and θ_k is available from one term of (A-1) by transforming to polar coordinates:

$$p_1(R_k, \theta_k) = \frac{R_k}{2\pi\sigma^2} \exp\left[-\frac{R_k^2 - 2P_0 R_k \cos(\theta_k - \psi_0) + P_0^2}{2\sigma^2}\right], R_k > 0, |\theta_k| < \pi. \quad \text{(A-18)}$$

The PDF of each θ_k is obtained by integrating over R_k:

$$p_1(\theta_k) = \frac{1}{2\pi}\left[\exp\left(-\frac{P_0^2}{2\sigma^2}\right) + \frac{P_0}{\sigma}\cos(\theta_k - \psi_0)\exp\left(-\frac{P_0^2}{2\sigma^2}\sin^2(\theta_k - \psi_0)\right) \cdot \int_{-\infty}^{\frac{P_0}{\sigma}\cos(\theta_k - \psi_0)} du \exp(-u^2/2)\right], |\theta_k| < \pi. \quad \text{(A-19)}$$

Now, letting $C_k = \cos(\theta_k - \psi_0)$ for notational simplicity, we expand

$$p_1(\theta_k) = \frac{1}{2\pi}\left[1 + \sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}C_k + \frac{P_0^2}{\sigma^2}\left(C_k^2 - \frac{1}{2}\right) + O\left(\frac{P_0^3}{\sigma^3}\right)\right], |\theta_k| < \pi, \quad \text{(A-20)}$$

where $O(\ )$ denotes terms the order of ( ) .

On the other hand, consider the approximation to $p_1(\theta_k)$ of the form

$$p_A(\theta_k) \equiv \frac{1}{2\pi}\exp\left[\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}C_k\right] \Big/ I_0\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\right), |\theta_k| < \pi. \quad \text{(A-21)}$$

This approximation has unit area and the expansion

$$p_A(\theta_k) = \frac{1}{2\pi}\left[1 + \sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}C_k + \frac{\pi}{4}\frac{P_0^2}{\sigma^2}\left(C_k^2 - \frac{1}{2}\right) + O\left(\frac{P_0^3}{\sigma^3}\right)\right], |\theta_k| < \pi. \quad \text{(A-22)}$$

Now, (A-20) and (A-22) are identical through order P_0/σ, and are almost equal through order P_0^2/σ^2 , the factor 1 in (A-20) being replaced by $\pi/4$ in (A-22). Therefore, we use the approximation (A-21) from this point on. Mathematically, we write

$$p_1(\theta_k) = \frac{1}{2\pi}\exp\left[\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}C_k\right] \Big/ I_0\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\right), \text{ to order } \frac{P_0}{\sigma}, |\theta_k| < \pi, \quad \text{(A-23)}$$

while remembering that (A-23) is also a good approximation to order P_0^2/σ^2 .

The joint PDF of $\{\theta_k\}$ is then given by

$$p_1(\underline{\theta}) = \frac{1}{(2\pi)^M} \exp\left[\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\sum_{k=1}^{M} C_k\right] \Big/ I_0^M\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\right), \text{ to order } \frac{P_0}{\sigma}, \; |\theta_k| < \pi. \tag{A-24}$$

The PDF of $\underline{\theta}$ for signal absent follows from (A-24) by setting $P_0 = 0$. The LR, to order P_0/σ, is therefore given by (A-24), with the factor $(2\pi)^{-M}$ absent. Thus, the LR test is

$$\sum_{k=1}^{M} C_k = \mathrm{Re}\left\{e^{-i\psi_0}\sum_{k=1}^{M} e^{i\theta_k}\right\} \gtrless T, \text{ to order } \frac{P_0}{\sigma}, \tag{A-25}$$

and is independent of P_0 and σ. Therefore, threshold T can be selected once and for all to realize a specified false alarm probability, P_F. The value of T depends on only M and P_F, and Processor IV is a CFAR receiver; i.e., test (A-25) is uniformly most powerful with respect to P_0 and σ, to order P_0/σ.

A.1.5 PROCESSOR V: UNKNOWN SIGNAL PHASE; PHASE SAMPLES

The PDF of $\underline{\theta}$ for unknown signal phase is given by (A-24), where ψ_0 in C_k is replaced by hypothesized angle ψ; i.e.,

$$p_1(\underline{\theta}|\psi) = \frac{1}{(2\pi)^M} \exp\left[\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\sum_{k=1}^{M}\cos(\theta_k - \psi)\right] \Big/ I_0^M\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\right), \quad \text{to order } \frac{P_0}{\sigma}, \; |\theta_k| < \pi. \tag{A-26}$$

We now express

$$\sum_{k=1}^{M}\cos(\theta_k - \psi) = \mathrm{Re}\left\{e^{-i\psi}\sum_{k=1}^{M} e^{i\theta_k}\right\}. \tag{A-27}$$

Then, for an a priori PDF of phase ψ that is uniform over 2π, the PDF $p_1(\underline{\theta})$ is given by

$$p_1(\underline{\theta}) = \frac{1}{2\pi}\int_{2\pi} d\psi \; p_1(\underline{\theta}|\psi)$$

$$= \frac{1}{(2\pi)^M} I_0\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\left|\sum_{k=1}^{M} e^{i\theta_k}\right|\right) \Big/ I_0^M\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\right), \text{ to order } \frac{P_0}{\sigma}, \; |\theta_k| < \pi. \tag{A-28}$$

Since the PDF of $\underline{\theta}$ for signal absent is obtained from (A-28) by setting $P_0 = 0$, the average LR is given by (A-28) without the factor $(2\pi)^{-M}$. Therefore the LR test is

$$\left|\sum_{k=1}^{M} e^{i\theta_k}\right| \gtrless T_1 \text{ to order } \frac{P_0}{\sigma}. \tag{A-29}$$

If, instead of assuming a uniform PDF for ψ, we choose ψ as the ML estimate in (A-26), we find, using (A-27), that the ML estimate is

$$\hat{\psi} = \arg\left\{\sum_{k=1}^{M} e^{i\theta_k}\right\} \tag{A-30}$$

and the maximum value of p_1 is

$$p_1(\underline{\theta}|\hat{\psi}) = \frac{1}{(2\pi)^M} \exp\left[\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\left|\sum_{k=1}^{M} e^{i\theta_k}\right|\right] \Big/ I_0^M\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\right), \tag{A-31}$$
$$\text{to order } \frac{P_0}{\sigma},\ |\theta_k| < \pi.$$

The generalized LR is the ratio of (A-31) to $(2\pi)^{-M}$ and, again, the generalized LR test is obviously (A-29). Thus, to order P_0/σ, two different methods of treating signal phase yield the same processor. Test (A-29) can be designed for specified P_F without knowledge of P_0 or σ.

A.1.6 PROCESSOR VI: UNKNOWN SIGNAL PHASE; FITTED PHASE SAMPLES

Instead of attempting to derive average LR or ML tests, we consider here a heuristic approach to detection. Specifically, the M successive $\{\theta_k\}$ are fitted by the best straight line such that the average squared-error is minimized. Then the resultant minimum error, or scatter, is used as a decision variable. For small SNR, the phase samples would be widely scattered over a 2π interval, whereas they would tend to cluster around the true signal phase for large SNR, yielding little scatter. Thus if the scatter is <u>less</u> than a threshold, we declare the signal present.

For known frequency, the best straight line has zero slope, as can be seen in (5). Therefore the average squared-error for hypothesized phase ψ is

$$E = \frac{1}{M}\sum_{k=1}^{M}(\theta_k - \psi)^2. \tag{A-32}$$

This is minimized by the choice

$$\hat{\psi} = \frac{1}{M}\sum_{k=1}^{M}\theta_k, \tag{A-33}$$

which is the average phase. The minimum error is therefore

$$\hat{E} = \frac{1}{M}\sum_{k=1}^{M}\theta_k^2 - \left(\frac{1}{M}\sum_{k=1}^{M}\theta_k\right)^2 . \tag{A-34}$$

When $\hat{E} < T$, the signal is declared present. Test statistic (A-34) is independent of P_0 and σ and yields a CFAR processor; the samples need not be equally spaced in time.

There is a problem in test (A-34) as to deciding the 2π interval or band in which to choose each θ_k. The choice definitely affects detectability, the exact degree depending on the true (unknown) signal phase ψ_0. This problem is discussed fully in the main text.

A.1.7 PROCESSOR VII: UNKNOWN SIGNAL PHASE; PHASE-DIFFERENCE SAMPLES

No derivation is necessary for this case. The phase-difference samples $\theta_k - \theta_{k-1}$ are formed for $2 \le k \le M$. For known signal frequency, the phase differences tend to cluster around zero for high SNR, even though signal phase ψ_0 is unknown. Thus, a random walk similar to (A-25) is formed and the real part is compared with a threshold. The test is

$$\mathrm{Re}\left\{\sum_{k=2}^{M}\exp\left[i\left(\theta_k - \theta_{k-1}\right)\right]\right\} \gtrless T. \tag{A-35}$$

The samples need not be equi-spaced in time.

A.2 UNKNOWN SIGNAL FREQUENCY

It is unrealistic to assume that signal phase would be known at some time instant for unknown signal frequency. Accordingly, there are no analogs of Processors I and IV in this case.

The received waveform here is a slight modification of (3):

$$r(t) = \mathrm{Re}\left\{\exp\left(i2\pi(f_0+f_d)t\right)\left[P_0\exp(i\psi_0) + \underline{n}(t)\right]\right\} \tag{A-36}$$

for Doppler shift f_d. The complex envelope of the received process is

$$\exp\left(i2\pi f_d t\right)\left[P_0\exp(i\psi_0) + \underline{n}(t)\right]. \tag{A-37}$$

A sample of the complex envelope at time t_k is denoted by

$$R_k e^{i\theta_k} = x_k + i y_k = \exp(i2\pi f_d t_k)\left[P_0 \exp(i\psi_0) + \underline{n}(t_k)\right]$$
$$\equiv P_0 \exp(i2\pi f_d t_k + i\psi_0) + n_x(t_k) + i n_y(t_k). \qquad \text{(A-38)}$$

The statistics of the real component samples, $n_x(t_k)$ and $n_y(t_k)$, of the noise are available from the following (reference 3):

$$\overline{n_x(t)} + i\,\overline{n_y(t)} = \overline{\underline{n}(t)}\exp(i2\pi f_d t) = 0,$$
$$\overline{|n_x(t) + i n_y(t)|^2} = \overline{|\underline{n}(t)|^2} = 2\sigma^2, \qquad \text{(A-39)}$$
$$\overline{[n_x(t) + i n_y(t)]^2} = \overline{\underline{n}^2(t)}\exp(i4\pi f_d t) = 0.$$

Therefore,

$$\overline{n_x^2(t)} = \overline{n_y^2(t)} = \sigma^2,$$
$$\overline{n_x(t)\,n_y(t)} = 0. \qquad \text{(A-40)}$$

These noise statistics are identical to those for known signal frequency.

A.2.1 PROCESSOR I: NO ANALOG

A.2.2 PROCESSOR II: UNKNOWN SIGNAL PHASE; AMPLITUDE AND PHASE SAMPLES

For signal present with hypothesized phase ψ and frequency shift f, the PDF of samples x_k and y_k in (A-38) is, using (A-38) and (A-40),

$$p_1(x_k, y_k|\psi, f) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{(x_k - P_0\cos(2\pi f t_k + \psi))^2 + (y_k - P_0\sin(2\pi f t_k + \psi))^2}{2\sigma^2}\right]. \qquad \text{(A-41)}$$

The PDF of $\underset{\sim}{x}$ and $\underset{\sim}{y}$ in (A-1) is therefore

$$p_1(\underset{\sim}{x}, \underset{\sim}{y}|\psi, f) = \frac{1}{(2\pi\sigma^2)^M}\exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^{M} R_k^2 \right. \qquad \text{(A-42)}$$
$$\left. - \frac{MP_0^2}{2\sigma^2} + \frac{P_0}{\sigma^2}\mathrm{Re}\left\{e^{-i\psi}\sum_{k=1}^{M} R_k e^{i\theta_k} e^{-i2\pi f t_k}\right\}\right].$$

If the a priori PDF of ψ is uniform, there follows

$$p_1(\underline{x}, \underline{y}|f) = \frac{1}{(2\pi\sigma^2)^M} \exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^{M} R_k^2 - \frac{MP_0^2}{2\sigma^2}\right] I_0\left(\frac{P_0}{\sigma^2}\left|\sum_{k=1}^{M} R_k e^{i\theta_k} e^{-i2\pi f t_k}\right|\right). \quad \text{(A-43)}$$

If, at this point we make an assumption about a PDF for f, we must make some approximation to (A-43) so that the evaluation will be amenable. The one we adopt is identical to that made earlier, namely, small SNR. Then, the relevant integral that must be evaluated is

$$\frac{1}{f_2 - f_1}\int_{f_1}^{f_2} df \sum_{k,\ell=1}^{M} R_k R_\ell \exp\left[i(\theta_k - \theta_\ell) - i2\pi f(t_k - t_\ell)\right], \text{ to order } \frac{P_0^2}{\sigma^2}, \quad \text{(A-44)}$$

where (f_1, f_2) is the range of anticipated Doppler shifts, and the PDF is assumed to be uniform. Now, for a large range $f_2 - f_1$, the only terms in (A-44) that yield significant contributions are $k = \ell$:

$$\sum_{k=1}^{M} R_k^2, \text{ to order } \frac{P_0^2}{\sigma^2}. \quad \text{(A-45)}$$

Alternately, (A-44) again reduces to (A-45) for equi-spaced samples, $t_{k+1} - t_k = \Delta$, and a uniform PDF of f over a $1/\Delta$ Hz range (which is approximately the frequency separation between adjacent filters as discussed following (2)). Thus no use would be made of $\{\theta_k\}$ under either assumption for the PDF of f. This is a case of designing for the worst possible situation and, thereby, drastically degrading performance (see reference 3, page 180).

A better approach is to return to (A-43) and choose f as the ML estimate; i.e., choose $\hat{f}$ such that

$$\left|\sum_{k=1}^{M} R_k e^{i\theta_k} e^{-i2\pi\hat{f}t_k}\right| \geq \left|\sum_{k=1}^{M} R_k e^{i\theta_k} e^{-i2\pi f t_k}\right|, \text{ all } f \in (f_1, f_2), \quad \text{(A-46)}$$

where (f_1, f_2) is the allowed range of Doppler shifts. Then the comparison of $p_1(\underline{x}, \underline{y}|\hat{f})/p_0(\underline{x}, \underline{y})$ with a threshold is equivalent to the test

$$\max_{f_1 < f < f_2} \left|\sum_{k=1}^{M} R_k e^{i\theta_k} e^{-i2\pi f t_k}\right| \gtrless T. \quad \text{(A-47)}$$

Another approach is to use ML estimates for both ψ and f. From (A-42) we see that the ML estimate of ψ is

$$\hat{\psi} = \arg\left\{\sum_{k=1}^{M} R_k e^{i\theta_k} e^{-i2\pi f t_k}\right\}. \quad \text{(A-48)}$$

Substitution of (A-48) in (A-42) and the subsequent choice of f for a maximum of p_1 again yields (A-46); and the resultant generalized LR test is again (A-47). This test is independent of P_0 ; also, no assumption about small SNR is required to deduce (A-47) as an appropriate test.

A.2.3 PROCESSOR III: INDEPENDENT SIGNAL PHASES; AMPLITUDE AND PHASE SAMPLES

For independent signal phases, the PDF is a modified form of (A-41):

$$p_1(x_k, y_k | \psi_k, f) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2\sigma^2}\left\{x_k - P_0 \cos(2\pi f t_k + \psi_k)\right\}^2 - \frac{1}{2\sigma^2}\left\{y_k - P_0 \sin(2\pi f t_k + \psi_k)\right\}^2\right]. \tag{A-49}$$

Therefore,

$$p_1(\underset{\sim}{x}, \underset{\sim}{y} | \underset{\sim}{\psi}, f) = \frac{1}{(2\pi\sigma^2)^M} \exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^{M} R_k^2 - \frac{M P_0^2}{2\sigma^2} + \frac{P_0}{\sigma^2} \operatorname{Re}\left\{\sum_{k=1}^{M} R_k \exp(i\theta_k - i\psi_k - i2\pi f t_k)\right\}\right]. \tag{A-50}$$

Now, if the phases are uniformly distributed over 2π,

$$p_1(\underset{\sim}{x}, \underset{\sim}{y} | f) = \frac{1}{(2\pi\sigma^2)^M} \exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^{M} R_k^2 - \frac{M P_0^2}{2\sigma^2}\right] \prod_{k=1}^{M} I_0\left(\frac{P_0}{\sigma^2} R_k\right). \tag{A-51}$$

Since this PDF is independent of f , a decent processor can not possibly result and we discard this approach.

If we instead take the ML estimates for $\underset{\sim}{\psi}$, we find from (A-50) that

$$\hat{\psi}_k = \arg\left\{\exp(i\theta_k - i2\pi f t_k)\right\}. \tag{A-52}$$

Then

$$p_1(\underset{\sim}{x}, \underset{\sim}{y} | \hat{\underset{\sim}{\psi}}, f) = \frac{1}{(2\pi\sigma^2)^M} \exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^{M} R_k^2 - \frac{M P_0^2}{2\sigma^2} + \frac{P_0}{\sigma^2}\sum_{k=1}^{M} R_k\right]. \tag{A-53}$$

Again, the independence of f yields an undesirable processor. Notice from (A-53) that the generalized LR test would take the form

$$\sum_{k=1}^{M} R_k \gtrless T. \tag{A-54}$$

A.2.4 PROCESSOR IV: NO ANALOG

A.2.5 PROCESSOR V: UNKNOWN SIGNAL PHASE; PHASE SAMPLES

Our starting point is (A-41). By transforming to polar coordinates according to (5), we obtain

$$p_1(R_k,\theta_k|\psi,f)=\frac{R_k}{2\pi\sigma^2}\exp\left[-\frac{1}{2\sigma^2}\left\{R_k^2+P_0^2-2P_0R_k\cos(\theta_k-\psi-2\pi f t_k)\right\}\right]. \quad \text{(A-55)}$$

Then

$$p_1(\theta_k|\psi,f)=\frac{1}{2\pi}\exp\left[\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\cos(\theta_k-\psi-2\pi f t_k)\right]\Big/I_0\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\right), \text{ to order } \frac{P_0}{\sigma}, \quad \text{(A-56)}$$

by means of an approach that is analogous to that in (A-18) through (A-23). Therefore

$$p_1(\underline{\theta}|\psi,f)=\frac{1}{(2\pi)^M}\exp\left[\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\operatorname{Re}\left\{e^{-i\psi}\sum_{k=1}^{M}e^{i\theta_k}e^{-i2\pi f t_k}\right\}\right]\times \quad \text{(A-57)}$$

$$I_0^{-M}\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\right), \text{ to order } \frac{P_0}{\sigma}.$$

If we assume that the PDF for ψ is uniform, we obtain

$$p_1(\underline{\theta}|f)=\frac{1}{(2\pi)^M}I_0\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\left|\sum_{k=1}^{M}\exp(i\theta_k-i2\pi f t_k)\right|\right)\times \quad \text{(A-58)}$$

$$I_0^{-M}\left(\sqrt{\frac{\pi}{2}}\frac{P_0}{\sigma}\right), \text{ to order } \frac{P_0}{\sigma}.$$

At this point, an assumption about the PDF of f leads us to the same problems encountered in (A-43) through (A-45), namely, undesirable processor forms. Accordingly, we choose the ML estimate $\hat{f}$ in (A-58):

$$\left|\sum_{k=1}^{M}\exp(i\theta_k-i2\pi\hat{f}t_k)\right|\geq\left|\sum_{k=1}^{M}\exp(i\theta_k-i2\pi f t_k)\right|, \text{ all } f\in(f_1,f_2), \quad \text{(A-59)}$$

and the generalized LR test then takes the form

$$\max_{f_1<f<f_2}\left|\sum_{k=1}^{M}\exp(i\theta_k-i2\pi f t_k)\right| \gtrless T, \text{ to order } \frac{P_0}{\sigma}. \quad \text{(A-60)}$$

The alternate approach of using ML estimates for both ψ and f in (A-57) is also possible: we find that

$$\hat{\psi}=\arg\left\{\sum_{k=1}^{M}\exp(i\theta_k-i2\pi f t_k)\right\}, \text{ to order } \frac{P_0}{\sigma}. \quad \text{(A-61)}$$

Substitution of (A-61) in (A-57) and the subseqnent choice of $\hat{f}$ again leads to (A-59) and the generalized LR test (A-60). This test is independent of P_0, but is accurate only to order P_0/σ.

A.2.6 PROCESSOR VI: UNKNOWN SIGNAL PHASE; FITTED PHASE SAMPLES

The philosophy here has been explained earlier in this appendix. However, the average squared-error for unknown frequency is generalized from (A-32) to

$$E = \frac{1}{M}\sum_{k=1}^{M}\left(\theta_k - \psi - \beta k\right)^2, \tag{A-62}$$

where we assume that the phase samples are equi-spaced in time. The partial derivatives of E with respect to ψ and β are both set equal tc zero and solved to obtain the estimates

$$\hat{\beta} = \frac{12}{M(M^2-1)}\sum_{k=1}^{M}\left(k - \frac{M+1}{2}\right)\theta_k,$$
$$\hat{\psi} = \frac{1}{M}\sum_{k=1}^{M}\theta_k - \frac{6}{M(M-1)}\sum_{k=1}^{M}\left(k - \frac{M+1}{2}\right)\theta_k. \tag{A-63}$$

Substitution of the estimates (A-63) in (A-62) yield the minimum scatter

$$\hat{E} = \frac{1}{M}\sum_{k=1}^{M}\theta_k^2 - \left(\frac{1}{M}\sum_{k=1}^{M}\theta_k\right)^2 - \frac{12}{M^2-1}\left(\frac{1}{M}\sum_{k=1}^{M}\left(k - \frac{M+1}{2}\right)\theta_k\right)^2. \tag{A-64}$$

When $\hat{E} < T$, the signal is declared present. Test (A-64) is independent of P_0 and is a CFAR processor.

A.2.7 PROCESSOR VII: UNKNOWN SIGNAL PHASE; PHASE-DIFFERENCE SAMPLES

For unknown signal frequency and equi-spaced samples, the phase differences $\theta_k - \theta_{k-1}$ cluster around an unknown angle for large SNR. The value of the unknown angle depends on the signal frequency and the time between samples. A random walk and magnitude similar to (A-29) is formed since only the length, and not the direction, indicates signal presence. Thus the test is

$$\left|\sum_{k=2}^{M}\exp\left(i\,\theta_k - i\,\theta_{k-1}\right)\right| \gtrless T. \tag{A-65}$$

Again, the test is independent of P_0 and yields a CFAR processor.

Appendix B

GEOMETRICAL INTERPRETATIONS OF PROCESSORS

No assumptions about Gaussian noise or independent samples are made in this appendix. Rather, geometrical interpretations of "good" processors are developed and the errors associated with each are minimized. The resultant tests are identical to those derived in appendix A. Although signal amplitude P_0 is assumed known, it is never needed in the tests.

B. 1 KNOWN SIGNAL FREQUENCY

B. 1. 1 PROCESSOR I: KNOWN SIGNAL PHASE; AMPLITUDE AND PHASE SAMPLES

If a signal is present, the complex samples $R_k e^{i\theta_k}$ should cluster about the point $P_0 e^{i\psi_0}$. A measure of the scatter is afforded by the sum of the squared distances between these points:

$$E_1 \equiv \sum_{k=1}^{M} \left| R_k e^{i\theta_k} - P_0 e^{i\psi_0} \right|^2$$

$$= \sum_{k=1}^{M} R_k^2 + M P_0^2 - 2P_0 \operatorname{Re}\left\{ e^{-i\psi_0} \sum_{k=1}^{M} R_k e^{i\theta_k} \right\}. \tag{B-1}$$

If, on the other hand, the signal is absent, the complex samples should cluster about the point 0. The scatter is then

$$E_0 \equiv \sum_{k=1}^{M} \left| R_k e^{i\theta_k} - 0 \right|^2 = \sum_{k=1}^{M} R_k^2 . \tag{B-2}$$

Now, if $E_1 < E_0$ by a sufficient amount, we would be quite sure that a signal is present. If, on the other hand, the converse is true, we would declare that the signal is absent. Therefore, the test we adopt is*

$$E_1 \lessgtr E_0 - A. \tag{B-3}$$

Substitution of (B-1) and (B-2) in (B-3) yields

$$\operatorname{Re}\left\{ e^{-i\psi_0} \sum_{k=1}^{M} R_k e^{i\theta_k} \right\} \gtrless T, \tag{B-4}$$

where T is a threshold that is adjusted for a prescribed P_F . Test (B-4) is identical to (A-4).

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*Here, as in the remainder of this appendix, satisfaction of the upper inequality leads to a statement of signal presence, whereas satisfaction of the lower inequality leads to a signal-absent decision.

B. 1. 2 PROCESSOR II: UNKNOWN SIGNAL PHASE; AMPLITUDE AND PHASE SAMPLES

For a hypothesized signal phase, ψ , the scatter

$$E_1 = \sum_{K=1}^{M} \left| R_K e^{i\theta_K} - P_0 e^{i\psi} \right|^2$$

$$= \sum_{K=1}^{M} R_K^2 + M P_0^2 - 2 P_0 \, \mathrm{Re}\left\{ e^{-i\psi} \sum_{K=1}^{M} R_K e^{i\theta_K} \right\}. \tag{B-5}$$

We can select ψ so that the scatter is minimized; i. e., we try to find that complex point $P_0 \exp(i\psi)$ about which the samples $R_K e^{i\theta_K}$ cluster best if a signal is present. The value of ψ that minimizes (B-5) is

$$\hat{\psi} = \arg\left\{ \sum_{K=1}^{M} R_K e^{i\theta_K} \right\}. \tag{B-6}$$

(Note that this is identical to the ML estimate A-10.) Then

$$E_1 = \sum_{K=1}^{M} R_K^2 + M P_0^2 - 2 P_0 \left| \sum_{K=1}^{M} R_K e^{i\theta_K} \right|. \tag{B-7}$$

For signal absent, E_0 is again given by (B-2). Substitution in (B-3) yields

$$\left| \sum_{K=1}^{M} R_K e^{i\theta_K} \right| \gtrless T, \tag{B-3}$$

which is identical to (A-8).

B. 1. 3 PROCESSOR III: INDEPENDENT SIGNAL PHASES; AMPLITUDE AND PHASE SAMPLES

For hypothesized signal phase ψ_K in sample K , the appropriate measure of scatter for signal present is

$$E_1 = \sum_{K=1}^{M} \left| R_K e^{i\theta_K} - P_0 e^{i\psi_K} \right|^2$$

$$= \sum_{K=1}^{M} R_K^2 + M P_0^2 - 2 P_0 \, \mathrm{Re}\left\{ \sum_{K=1}^{M} R_K e^{i(\theta_K - \psi_K)} \right\}. \tag{B-9}$$

Since we must be allowed to minimize E_1 by choice of ψ_K , we choose

$$\hat{\psi}_K = \theta_K, \tag{B-10}$$

yielding

$$E_1 = \sum_{K=1}^{M} R_K^2 + M P_0^2 - 2 P_0 \sum_{K=1}^{M} R_K . \tag{B-11}$$

Scatter E_0 is given by (B-2). Test (B-3) then yields

$$\sum_{K=1}^{M} R_K \gtrless T. \qquad \text{(B-12)}$$

Test (B-12) is not identical to LR test (A-16) or to the approximate LR test (A-17) that was derived for small SNR. However, if the SNR is large $(P_0/\sigma \gg 1)$, (A-16) tends to (B-12) and, therefore, (B-12) is nearly optimum for high SNR. Test (B-12) is not, however, pursued further in this report.

B. 1. 4 PROCESSOR IV: KNOWN SIGNAL PHASE; PHASE SAMPLES

Since amplitude samples $\{R_K\}$ are not used, it is expected that complex samples $\{\exp(i\theta_K)\}$ would cluster around the point $\exp(i\psi_0)$ for large SNR. Therefore, a meaningful measure of scatter is given by

$$E_1 = \sum_{K=1}^{M} \left| e^{i\theta_K} - e^{i\psi_0} \right|^2$$
$$= 2M - 2\,\mathrm{Re}\left\{ e^{-i\psi_0} \sum_{K=1}^{M} e^{i\theta_K} \right\}. \qquad \text{(B-13)}$$

Scatter E_0 is now given by

$$E_0 = \sum_{K=1}^{M} \left| e^{i\theta_K} - 0 \right|^2 = M. \qquad \text{(B-14)}$$

Thus, evaluation of (B-3) yields

$$\mathrm{Re}\left\{ e^{-i\psi_0} \sum_{K=1}^{M} e^{i\theta_K} \right\} \gtrless T, \qquad \text{(B-15)}$$

which is identical to (A-25).

B. 1. 5 PROCESSOR V: UNKNOWN SIGNAL PHASE; PHASE SAMPLES

For hypothesized signal phase ψ, the scatter for signal present is

$$E_1 = \sum_{K=1}^{M} \left| e^{i\theta_K} - e^{i\psi} \right|^2$$
$$= 2M - 2\,\mathrm{Re}\left\{ e^{-i\psi} \sum_{K=1}^{M} e^{i\theta_K} \right\}. \qquad \text{(B-16)}$$

This is minimized by the choice

$$\hat{\psi} = \arg\left\{ \sum_{K=1}^{M} e^{i\theta_K} \right\}, \qquad \text{(B-17)}$$

which is identical to ML estimate (A-30), and leads to

$$E_1 = 2M - 2\left|\sum_{K=1}^{M} e^{i\theta_K}\right|. \qquad \text{(B-18)}$$

Scatter E_0 is given by (B-14) and substitution in (E-3) yields

$$\left|\sum_{K=1}^{M} e^{i\theta_K}\right| \gtrless T, \qquad \text{(B-19)}$$

which is identical to (A-29).

B.2 UNKNOWN SIGNAL FREQUENCY

It is unrealistic to assume that signal phase would be known at some time instant for unknown signal frequency. Accordingly, there are no analogs of Processors I and IV in this case.

B.2.1 PROCESSOR I: NO ANALOG

B.2.2 PROCESSOR II: UNKNOWN SIGNAL PHASE; AMPLITUDE AND PHASE SAMPLES

For high SNR, the complex samples $\{R_K e^{i\theta_K}\}$ in (5) should cluster around a uniformly rotating complex vector, $P_0 \exp(i\psi + i2\pi ft)$, where initial phase ψ and rotation rate f are unknown. The measure of scatter adopted is therefore

$$\begin{aligned} E_1 &= \sum_{K=1}^{M} \left| R_K e^{i\theta_K} - P_0 \exp(i\psi + i2\pi f t_K)\right|^2 \\ &= \sum_{K=1}^{M} R_K^2 + MP_0^2 - 2P_0 \,\mathrm{Re}\left\{e^{-i\psi} \sum_{K=1}^{M} R_K e^{i\theta_K} e^{-i2\pi f t_K}\right\}. \end{aligned} \qquad \text{(B-20)}$$

This is minimized by the choice of hypothesized signal phase as

$$\hat{\psi} = \arg\left\{\sum_{K=1}^{M} R_K e^{i\theta_K} e^{-i2\pi f t_K}\right\}, \qquad \text{(B-21)}$$

which is identical to (A-48). The resultant value of E_1,

$$E_1 = \sum_{K=1}^{M} R_K^2 + MP_0^2 - 2P_0\left|\sum_{K=1}^{M} R_K e^{i\theta_K} e^{-i2\pi f t_K}\right|, \qquad \text{(B-22)}$$

is further minimized by choosing f such that the last term in (B-22) is maximized; i.e., choose $\hat{f}$ such that

$$\left|\sum_{k=1}^{M} R_k e^{i\theta_k} e^{-i2\pi\hat{f}t_k}\right| \geq \left|\sum_{k}^{M} R_k e^{i\theta_k} e^{-i2\pi f t_k}\right| \text{ for all } f \in (f_1, f_2). \tag{B-23}$$

This is identical to (A-46). Then

$$E_1 = \sum_{k=1}^{M} R_k^2 + M P_o^2 - 2P_o \max_{f_1<f<f_2} \left|\sum_{k=1}^{M} R_k e^{i\theta_k} e^{-i2\pi f t_k}\right|. \tag{B-24}$$

The value of scatter E_o for signal absent is still given by (B-2). Therefore, (B-3) yields

$$\max_{f_1<f<f_2} \left|\sum_{k=1}^{M} R_k e^{i\theta_k} e^{-i2\pi f t_k}\right| \gtrless T_1 \tag{B-25}$$

which is identical to (A-47).

B. 2. 3 PROCESSOR III: INDEPENDENT SIGNAL PHASES; AMPLITUDE AND PHASE SAMPLES

The appropriate measure of spread for independent phases is

$$E_1 = \sum_{k=1}^{M} \left|R_k e^{i\theta_k} - P_o e^{i\psi_k}\right|^2. \tag{B-26}$$

Since this is the same measure that was used for known frequency in (B-9), test (B-12) is not an acceptable test for unknown frequency; it is, however, identical to (A-54).

B. 2. 4 PROCESSOR IV: NO ANALOG

B. 2. 5 PROCESSOR V: UNKNOWN SIGNAL PHASE; PHASE SAMPLES

The appropriate scatter here is a modification of (B-20):

$$\begin{aligned} E_1 &= \sum_{k=1}^{M} \left|e^{i\theta_k} - e^{i\psi + i2\pi f t_k}\right|^2 \\ &= 2M - 2\,\mathrm{Re}\left\{e^{-i\psi} \sum_{k=1}^{M} e^{i\theta_k} e^{-i2\pi f t_k}\right\}. \end{aligned} \tag{B-27}$$

This scatter is minimized by the choice of hypothesized signal phase

$$\hat{\psi} = \arg\left\{\sum_{K=1}^{M} \exp(i\theta_K - i2\pi f t_K)\right\}, \tag{B-28}$$

which is identical to (A-61). Then

$$E_1 = 2M - 2\left|\sum_{K=1}^{M} \exp(i\theta_K - i2\pi f t_K)\right| \tag{B-29}$$

can be further minimized by choosing $\hat{f}$ such that

$$\left|\sum_{K=1}^{M} e^{i\theta_K} e^{-i2\pi\hat{f}t_K}\right| \geq \left|\sum_{K=1}^{M} e^{i\theta_K} e^{-i2\pi f t_K}\right| \text{ for all } f \in (f_1, f_2), \tag{B-30}$$

which is identical to (A-59). Then

$$E_1 = 2M - 2 \max_{f_1<f<f_2} \left|\sum_{K=1}^{M} \exp(i\theta_K - i2\pi f t_K)\right|. \tag{B-31}$$

The value of scatter E_0 for signal absent is still given by (B-14). Therefore, (B-3) yields

$$\max_{f_1<f<f_2} \left|\sum_{K=1}^{M} \exp(i\theta_K - i2\pi f t_K)\right| \gtrless T, \tag{B-32}$$

which is identical to (A-60).

Appendix C

DERIVATIONS OF PROCESSOR PERFORMANCE

The additive noise is assumed to be Gaussian, and the samples statistically independent as noted in the discussion following (2). Derivations of the detection probability, P_D, will be made or approximated where possible, and numerical methods that would lead to exact performance calculations in some cases will be outlined. Such situations were not pursued to completion in all cases, however.

Some of the derivations to follow are condensed in the interest of brevity. However, the essential steps are presented in such a way that the reader can follow them and insert the missing calculations.

C.1 KNOWN SIGNAL FREQUENCY

C.1.1 PROCESSOR I: KNOWN SIGNAL PHASE; AMPLITUDE AND PHASE SAMPLES

From table 1 and equation (5), we express the decision variable as

$$\ell = \cos\psi_0 \sum_{K=1}^{M} x_K + \sin\psi_0 \sum_{K=1}^{M} y_K . \qquad \text{(C-1)}$$

Random variables (RV) $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are Gaussian, with the PDF given by (A-2) for signal present. Then RV $\underset{\sim}{\ell}$ is Gaussian, with

$$\begin{aligned} E(\ell) &= \cos\psi_0\, M P_0 \cos\psi_0 + \sin\psi_0\, M P_0 \sin\psi_0 = M P_0 , \\ \mathrm{Var}(\ell) &= \cos^2\psi_0\, M\, \mathrm{Var}(x) + \sin^2\psi_0\, M\, \mathrm{Var}(x) = M\sigma^2 , \end{aligned} \qquad \text{(C-2)}$$

where we used the independence of all the RVs. Therefore

$$\begin{aligned} P_D = \mathrm{Prob}(\ell > T) &= \int_T^{\infty} d\ell \left(2\pi M\sigma^2\right)^{-1/2} \exp\left[-\frac{1}{2M\sigma^2}\left(\ell - MP_0\right)^2\right] \\ &= \Phi\left(\sqrt{M}\,\frac{P_0}{\sigma} - \frac{T}{\sqrt{M}\,\sigma}\right), \end{aligned} \qquad \text{(C-3)}$$

where

$$\Phi(x) \equiv \int_{-\infty}^{x} dy\, (2\pi)^{-1/2} \exp(-y^2/2) \equiv \int_{-\infty}^{x} dy\, \phi(y). \qquad \text{(C-4)}$$

If we define a normalized threshold $\lambda = \frac{T}{\sqrt{M}\,\sigma}$ and set $m = \frac{P_o}{\sigma}$, it follows from (C-3) that

$$P_D = \Phi\left(\sqrt{M}\,m - \lambda\right), \quad P_F = \Phi(-\lambda); \tag{C-5}$$

false alarm probability P_F was obtained by setting $m = 0$ in P_D.

C.1.2 PROCESSOR II: UNKNOWN SIGNAL PHASE; AMPLITUDE AND PHASE SAMPLES

From table 1 and (5),

$$\ell = \left[\left(\sum_{k=1}^{M} x_k\right)^2 + \left(\sum_{k=1}^{M} y_k\right)^2\right]^{1/2} \equiv \left[u^2 + v^2\right]^{1/2}. \tag{C-6}$$

The PDF of $\underset{\sim}{x}$ and $\underset{\sim}{y}$ is given by (A-2), and since u and v are Gaussian RVs, we have

$$\text{Prob}(\ell > T) = \iint\limits_{\sqrt{u^2+v^2}>T} du\, dv\; p(u,v)$$

$$= \iint\limits_{\sqrt{u^2+v^2}>T} du\, dv\, \left(2\pi M\sigma^2\right)^{-1} \exp\left[-\frac{1}{2M\sigma^2}\left\{\left(u - MP_o\cos\psi_o\right)^2 + \left(v - MP_o\sin\psi_o\right)^2\right\}\right]$$

$$= \int_T^\infty dr\, r \int_{2\pi} d\theta\, \left(2\pi M\sigma^2\right)^{-1} \exp\left[-\frac{1}{2M\sigma^2}\left\{r^2 + M^2P_o^2 - 2MP_o r\cos(\theta - \psi_o)\right\}\right] \tag{C-7}$$

$$= \int_T^\infty dr\, \frac{r}{M\sigma^2} \exp\left[-\frac{r^2 + M^2P_o^2}{2M\sigma^2}\right] I_o\left(\frac{P_o r}{\sigma^2}\right)$$

$$= Q\left(\sqrt{M}\,\frac{P_o}{\sigma}, \frac{T}{\sqrt{M}\,\sigma}\right) = Q\left(\sqrt{M}\,m, \lambda\right),$$

independent of ψ_o, where (reference 3, appendix F)

$$Q(a,b) = \int_b^\infty dx\; x\, \exp\left(-\frac{x^2 + a^2}{2}\right) I_o(ax). \tag{C-8}$$

Therefore

$$P_D = Q\left(\sqrt{M}\,m, \lambda\right), \quad P_F = Q(0, \lambda) = \exp\left(-\lambda^2/2\right). \tag{C-9}$$

C. 1. 3 PROCESSOR III: INDEPENDENT SIGNAL PHASES; AMPLITUDE AND PHASE SAMPLES

From table 1 and (5),

$$\ell = \sum_{k=1}^{M} (x_k^2 + y_k^2). \tag{C-10}$$

The characteristic function (CF) of RV ℓ is

$$f(\xi) = E\{\exp(i\xi\ell)\} = \prod_{k=1}^{M} E\{\exp[i\xi(x_k^2 + y_k^2)]\}$$

$$= \left\{\iint dx\, dy\, (2\pi\sigma^2)^{-1} \exp\left[i\xi(x^2+y^2) - \frac{1}{2\sigma^2}\left\{(x-P_o\cos\psi_o)^2 + (y-P_o\sin\psi_o)^2\right\}\right]\right\}^M \tag{C-11}$$

$$= (1-i2\sigma^2\xi)^{-M} \exp\left[\frac{MP_o^2}{\sigma^2}\,\frac{i\sigma^2\xi}{1-i2\sigma^2\xi}\right],$$

upon transforming to polar coordinates and using reference 6, 6.631 4. The probabilities of detection and false alarm are then (reference 3, pp. 217-219)

$$P_D = Q_M\left(\sqrt{M}\,m, \lambda\right), \quad P_F = Q_M(0, \lambda), \tag{C-12}$$

where

$$Q_M(a,b) = \int_b^\infty dx\, x \left(\frac{x}{a}\right)^{M-1} \exp\left(-\frac{x^2+a^2}{2}\right) I_{M-1}(ax)$$

$$= Q(a,b) + \exp\left(-\frac{a^2+b^2}{2}\right) \sum_{r=1}^{M-1} \left(\frac{b}{a}\right)^r I_r(ab), \tag{C-13}$$

and, in particular,

$$P_F = Q_M(0, \lambda) = \exp(-\lambda^2/2) \sum_{r=0}^{M-1} \frac{1}{r!}\left(\frac{\lambda^2}{2}\right)^r \tag{C-14}$$

Extensive numerical results are provided in reference 5.

C. 1. 4 PROCESSOR IV: KNOWN SIGNAL PHASE; PHASE SAMPLES

It is convenient at this point to make a change of variables; using (5),

$$(x_k + i y_k)\exp(-i\psi_o) = R_k \exp(i\theta_k - i\psi_o) = P_o + \underline{n}(t_k)e^{-i\psi_o} \equiv P_o + \sigma(a_k + i b_k). \tag{C-15}$$

That is,

$$a_K + i\, b_K = \frac{1}{\sigma} \exp(-i\psi_0)\, \underline{n}(t_K). \tag{C-16}$$

Then (reference 3 or 4)

$$E(a_K) + i\, E(b_K) = \frac{1}{\sigma} \exp(-i\psi_0)\, E(\underline{n}(t_K)) = 0,$$

$$E\left(|a_K + i b_K|^2\right) = E(a_K^2) + E(b_K^2) = \frac{1}{\sigma^2} E\left(|\underline{n}(t_K)|^2\right) = 2, \tag{C-17}$$

$$E\left((a_K + i b_K)^2\right) = E(a_K^2) - E(b_K^2) + i2E(a_K b_K) = \frac{1}{\sigma^2} e^{-i2\psi_0} E\left(\underline{n}^2(t_K)\right) = 0.$$

Therefore

$$\begin{aligned} E(a_K) &= E(b_K) = 0, \\ E(a_K b_K) &= 0, \\ E(a_K^2) &= E(b_K^2) = 1. \end{aligned} \tag{C-18}$$

Since (C-16) is a linear transformation on a complex Gaussian RV, (C-18) indicates that a_K and b_K are independent zero-mean unit-variance Gaussian RVs.

Then from table 1 and (C-15), the decision variable becomes

$$\ell = \mathrm{Re}\left\{ \sum_{K=1}^{M} \frac{P_0 + \sigma(a_K + i b_K)}{|P_0 + \sigma(a_K + i b_K)|} \right\} = \sum_{K=1}^{M} \frac{m + a_K}{\left[(m + a_K)^2 + b_K^2\right]^{1/2}}, \tag{C-19}$$

where

$$m \equiv \frac{P_0}{\sigma}. \tag{C-20}$$

<u>Exact Detection Probability</u>

The first order PDF of

$$s \equiv \frac{m + a}{\left[(m+a)^2 + b^2\right]^{1/2}} \tag{C-21}$$

can be evaluated in closed form as follows. Let $t = m + a$; then, for $|r| < 1$,

$$\mathrm{Prob}(s < r) = \mathrm{Prob}\left(\frac{t}{[t^2 + b^2]^{1/2}} < r\right) = \mathrm{Prob}\left(t < \frac{r}{\sqrt{1-r^2}}|b|\right)$$

$$= \int_{-\infty}^{\infty} db\, p(b) \int_{-\infty}^{\frac{r|b|}{\sqrt{1-r^2}}} dt\, p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} db\, \exp(-b^2/2) \int_{-\infty}^{\frac{r|b|}{\sqrt{1-r^2}}} dt\, \exp\left(-\tfrac{1}{2}(t-m)^2\right) = \int_{-\infty}^{\infty} db\, \phi(b)\, \Phi\left(\frac{r|b|}{\sqrt{1-r^2}} - m\right), \tag{C-22}$$

using (C-21), (C-18), and (C-4). Therefore, the PDF of s, for $|r| < 1$, is

$$p_s(r) = \frac{d}{dr}\,\text{Prob}(s < r) = \int_{-\infty}^{\infty} db\, \phi(b)\, \Phi'\left(\frac{r|b|}{\sqrt{1-r^2}} - m\right) \frac{d}{dr}\left\{\frac{r|b|}{\sqrt{1-r^2}}\right\}$$

$$= (1-r^2)^{-3/2} \int_{-\infty}^{\infty} db\, \phi(b)\, \phi\left(\frac{r|b|}{\sqrt{1-r^2}} - m\right)|b|$$

$$= 2(1-r^2)^{-3/2} \int_0^{\infty} db\, \phi(b)\, \phi\left(\frac{rb}{\sqrt{1-r^2}} - m\right) b \tag{C-23}$$

$$= \frac{1}{\pi\sqrt{1-r^2}}\left[\exp(-m^2/2) + \sqrt{2\pi}\, mr \exp\left(-(1-r^2)m^2/2\right) \Phi(mr)\right],$$

upon completing the square in the exponent.

Now the CF of s, which is given by

$$f_s(\xi) = \int_{-1}^{1} dr \exp(i\xi r)\, p_s(r), \tag{C-24}$$

can be quickly evaluated by a fast Fourier transform (FFT). (This CF is also available directly from (C-21) and (C-18) as

$$f_s(\xi) = \int_0^{\infty} dr\, r \exp\left(-\frac{r^2+m^2}{2}\right) I_0(mr + i\xi), \tag{C-25}$$

but that method is much more time-consuming.) Then, the CF of RV ℓ in (C-19) is $f_s^M(\xi)$. Finally, the cumulative distribution of RV ℓ is available (from reference 7, (7)) as

$$P_D = \text{Prob}(\ell > T) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \frac{d\xi}{\xi}\, \text{Im}\left\{f_s^M(\xi) \exp(-i\xi T)\right\}, \tag{C-26}$$

which can also be evaluated by an FFT (reference 8). This exact procedure has not been pursued in this report.

Exact False Alarm Probability

The CF of s for signal absent is given by (C-25) with $m = 0$:

$$f_s(\xi) = I_0(i\xi) = J_0(\xi). \tag{C-27}$$

Therefore the CF of RV ℓ in (C-19) is

$$f_\ell(\xi) = J_0^M(\xi), \tag{C-28}$$

and thus (reference 7, (7)),

$$P_F = \mathrm{Prob}(\ell > T) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{d\xi}{\xi} J_0^M(\xi) \sin(\xi T)$$

$$= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty dx \, \frac{\sin x}{x} J_0^M\left(\sqrt{\frac{2}{M}} \frac{x}{\lambda}\right), \tag{C-29}$$

where normalized threshold

$$\lambda \equiv \sqrt{\frac{2}{M}}\, T. \tag{C-30}$$

Equation (C-29) is an exact relation for P_F ; it is seen to depend on only λ and M. Therefore, Processor IV is a CFAR receiver; i. e., λ can be chosen to realize a prescribed P_F, requiring only knowledge of M. The reason for the threshold scaling in (C-29) and (C-30) is that

$$J_0^M\left(\sqrt{\frac{2}{M}}\, u\right) \sim \exp(-u^2/2) \quad \text{as } M \to \infty. \tag{C-31}$$

Thus, (C-29) yields (see reference 6, 3.896 4; integrate both sides with respect to b)

$$P_F \sim \Phi(-\lambda) \quad \text{as } M \to \infty. \tag{C-32}$$

Therefore it is anticipated that the choice of λ in the exact relation (C-29) for specified P_F becomes relatively independent of M, for large M. Table C-1 gives values of λ, as determined from (C-29), for $M = 25$.

Table C-1. Thresholds Required for Processor IV for M = 25; Known Signal Frequency

P_F	$\lambda(P_F, M = 25)$
10^{-1}	1.28597
10^{-2}	2.31209
10^{-3}	3.03814
10^{-4}	3.61396

Approximate Detection Probability

The RV ℓ in (C-19) becomes approximately Gaussian for large M, and we need evaluate only its mean and variance to approximate its PDF. From (C-24) and (C-25)

$$f_s'(0) = i\, E(s) = \int_0^\infty dr \, r \exp\left(-\frac{r^2+m^2}{2}\right) I_1(mr)\, i. \tag{C-33}$$

Therefore (reference 6, 6.618 4)

$$E(s) = \left(\frac{\pi}{8}\right)^{1/2} m \exp(-m^2/4)\left[I_0(m^2/4) + I_1(m^2/4)\right] \equiv m_s . \tag{C-34}$$

Also,

$$f_s''(0) = -E(s^2) = \int_0^\infty dr\, r \exp\left(-\frac{r^2+m^2}{2}\right) I_1'(mr)\,(-1), \tag{C-35}$$

yielding (reference 9, 9.6.26 and 10.2.13 and reference 6, 6.618 4)

$$E(s^2) = \int_0^\infty dr\, r \exp\left(-\frac{r^2+m^2}{2}\right)\left[I_0(mr) - \frac{I_1(mr)}{mr}\right]$$

$$= 1 - \frac{1}{m}\sqrt{\frac{\pi}{2}} \exp(-m^2/4)\, I_{1/2}(m^2/4) \tag{C-36}$$

$$= 1 - \frac{1}{m^2}\left[1 - \exp(-m^2/2)\right].$$

The variance of RV s is

$$\mathrm{Var}(s) = 1 - \frac{1}{m^2}\left[1 - \exp(-m^2/2)\right] - m_s^2 \equiv \sigma_s^2 , \tag{C-37}$$

using (C-36) and (C-34).

Then the PDF of RV ℓ in (C-19) is

$$p(\ell) \cong \left(2\pi M \sigma_s^2\right)^{-1/2} \exp\left[-\frac{1}{2M\sigma_s^2}\left(\ell - M m_s\right)^2\right] \tag{C-38}$$

and

$$\mathrm{Prob}(\ell > T) \cong \Phi\left(\sqrt{M}\,\frac{m_s}{\sigma_s} - \frac{T}{\sqrt{M}\,\sigma_s}\right). \tag{C-39}$$

Denoting the approximate detection probability by $\tilde{P}_D$ and using (C-30), we have

$$\tilde{P}_D = \Phi\left(\sqrt{M}\,\frac{m_s}{\sigma_s} - \frac{\lambda}{\sqrt{2}\,\sigma_s}\right), \tag{C-40}$$

where m_s and σ_s are given by (C-34) and (C-37), respectively. As the signal strength approaches zero, $m \to 0$, and (C-40) approaches the approximate false alarm probability, $\tilde{P}_F$:

$$\tilde{P}_F = \tilde{P}_D(m=0) = \Phi\left(-\frac{\lambda}{\sqrt{2}\,\sigma_s(m=0)}\right) = \Phi(-\lambda), \tag{C-41}$$

using (C-34) and (C-37). This result is identical to (C-32), which is consistent with the observation that both are valid for large M. Thus, (C-40) and (C-41) constitute the large-M approximation for detection and false alarm probabilities. For small m, the factor m_s/σ_s in (C-40) is approximately $\frac{1}{2}\sqrt{\pi}\,m$; comparison with (C-5) indicates that Processor IV is 1.05 dB poorer than Processor I for large M and small SNR, m.

To obtain exact receiver operating characteristics for this processor, (C-29) should first be solved for the required $\lambda = \lambda(P_F, M)$. Then, that value of λ, along with (C-30), should be used in the exact relation (C-26) to obtain P_D. If, instead, approximate values of P_D are acceptable, (C-40) should be used, where the values given above are used for $\lambda(P_F, M)$. That is, the values of λ used in (C-40) should not be obtained from the approximation (C-41), but from the exact relation (C-29).

C. 1. 5 PROCESSOR V: UNKNOWN SIGNAL PHASE; PHASE SAMPLES

From table 1 and equations (5), (C-15), and (C-20), the decision variable is

$$\ell = \left| \frac{1}{M} \sum_{k=1}^{M} \frac{m + a_k + i b_k}{\left[(m+a_k)^2 + b_k^2\right]^{1/2}} \right| \equiv |u + iv|. \tag{C-42}$$

A factor of $1/M$ has been supplied for convenience; it can be absorbed by threshold adjustment. We have not been able to derive a tractable method for evaluating the detection probability exactly, but an exact false alarm probability calculation is possible and a good approximation for the detection probability has been attained.

Exact False Alarm Probability

For $m = 0$, we have

$$\ell = |u + iv|, \quad u = \frac{1}{M} \sum_{k=1}^{M} \frac{a_k}{(a_k^2 + b_k^2)^{1/2}}, \quad v = \frac{1}{M} \sum_{k=1}^{M} \frac{b_k}{(a_k^2 + b_k^2)^{1/2}}. \tag{C-43}$$

The second-order CF of RVs u and v is

$$\begin{aligned} f(\xi, \eta) &= E\{\exp(i\xi u + i\eta v)\} \\ &= \left\{ \iint da\, db\, \exp\left[i \frac{\xi a + \eta b}{M(a^2 + b^2)^{1/2}}\right] \frac{1}{2\pi} \exp\left[-\frac{a^2 + b^2}{2}\right] \right\}^M \\ &= J_0^M\left(\frac{1}{M}\sqrt{\xi^2 + \eta^2}\right), \end{aligned} \tag{C-44}$$

using (C-18) and changing to polar coordinates. Then, the second-order PDF of u and v is

$$p(u,v) = \frac{1}{(2\pi)^2}\iint d\xi\, d\eta\, \exp(-i\xi u - i\eta v)\, J_0^M\left(\frac{1}{M}\sqrt{\xi^2+\eta^2}\right) \tag{C-45}$$
$$= \frac{1}{2\pi}\int_0^\infty dr\, r\, J_0\left(r\sqrt{u^2+v^2}\right) J_0^M(r/M).$$

Therefore

$$1 - P_F = \operatorname{Prob}(\ell < T) = \iint_{u^2+v^2<T^2} du\, dv\, \frac{1}{2\pi}\int_0^\infty dr\, r\, J_0\left(r\sqrt{u^2+v^2}\right) J_0^M(r/M) \tag{C-46}$$
$$= \int_0^\infty dx\, J_1(x)\, J_0^M\left(\frac{x}{MT}\right),$$

upon interchanging integrals, changing to polar coordinates, and employing reference 9, 9.1.30. Thus

$$P_F = 1 - \int_0^\infty dx\, J_1(x)\, J_0^M\left(\sqrt{\frac{2}{M}}\,\frac{x}{\lambda}\right), \tag{C-47}$$

where normalized threshold

$$\lambda \equiv \sqrt{2M}\, T. \tag{C-48}$$

Since P_F depends on only M and λ, Processor V is a CFAR receiver. The reason for the threshold scaling in (C-47) and (C-48) is that

$$P_F \sim 1 - \int_0^\infty dx\, J_1(x)\exp\left(-\frac{x^2}{2\lambda^2}\right) \quad \text{as} \quad M \to \infty, \tag{C-49}$$

or

$$P_F \sim \exp(-\lambda^2/2) \quad \text{as} \quad M \to \infty, \tag{C-50}$$

where we have employed (C-31) and reference 6, 6.618 1. Therefore the choice of $\lambda(P_F, M)$ in the exact relation (C-47) becomes relatively independent of M for large M. Table C-2 gives values of λ, as determined from (C-47), for $M=25$.

Table C-2. Thresholds Required for Processor V for M = 25; Known Signal Frequency

P_F	$\lambda(P_F, M = 25)$
10^{-1}	2.14272
10^{-2}	2.99424
10^{-3}	3.62192
10^{-4}	4.12829

Approximate Detection Probability

The RVs u and v in (C-42) tend toward Gaussian RVs for large M. We assume that the second-order PDF of u and v can be approximated by a joint Gaussian PDF. Therefore we must evaluate all first- and second-order moments of u and v. From (C-42) we have

$$u = \frac{1}{M}\sum_{k=1}^{M} \frac{m + a_k}{\left[(m+a_k)^2 + b_k^2\right]^{1/2}} \,, \quad v = \frac{1}{M}\sum_{k=1}^{M} \frac{b_k}{\left[(m+a_k)^2 + b_k^2\right]^{1/2}} \,. \qquad \text{(C-51)}$$

A comparison of (C-51) with (C-21) and the use of (C-34) shows that

$$E(u) = \left(\frac{\pi}{8}\right)^{1/2} m \exp(-m^2/4)\left[I_0(m^2/4) + I_1(m^2/4)\right] \equiv m_u \,. \qquad \text{(C-52)}$$

Also, using (C-18), it follows that

$$E(v) = \iint da\, db \frac{b}{\left[(m+a)^2 + b^2\right]^{1/2}} \frac{1}{2\pi} \exp\left(-\frac{a^2+b^2}{2}\right) = 0, \qquad \text{(C-53)}$$

since the integrand is odd in b.

The correlation between u and v is

$$E(uv) = \frac{1}{M^2}\sum_{k,\ell=1}^{M} E\left\{\frac{(m+a_k)\, b_\ell}{\left[(m+a_k)^2+b_k^2\right]^{1/2}\left[(m+a_\ell)^2+b_\ell^2\right]^{1/2}}\right\} = 0, \qquad \text{(C-54)}$$

because odd integrands cause all averages on RVs b_ℓ to be zero.

The variance of RV v is

$$\sigma_v^2 = E(v^2) = \frac{1}{M^2}\sum_{k,\ell=1}^{M} E\left\{\frac{b_k\, b_\ell}{\left[(m+a_k)^2+b_k^2\right]^{1/2}\left[(m+a_\ell)^2+b_\ell^2\right]^{1/2}}\right\} \qquad \text{(C-55)}$$

$$= \frac{1}{M} E\left\{\frac{b^2}{(m+a)^2+b^2}\right\} + \left(1-\frac{1}{M}\right) E^2\left\{\frac{b}{\left[(m+a)^2+b^2\right]^{1/2}}\right\},$$

where we have separated the $k=\ell$ terms from the $k\neq\ell$ terms. The second term in (C-55) is, according to (C-53), zero; and the first term, upon changing to polar coordinates, is

$$\sigma_v^2 = \frac{1}{M}\int_0^\infty dr\, r \int_{2\pi} d\theta \frac{1}{2\pi} \sin^2\theta \exp\left[-\frac{1}{2}\left(r^2 - 2mr\cos\theta + m^2\right)\right] \qquad \text{(C-56)}$$

$$= \frac{1}{M}\int_0^\infty dr \frac{1}{m} I_1(mr) \exp\left(-\frac{r^2+m^2}{2}\right) = \frac{1}{M}\,\frac{1-\exp(-m^2/2)}{m^2},$$

where we have used

$$\frac{1}{2\pi}\int_{2\pi} d\theta\, \sin^2\theta\, \exp(a\cos\theta) = \frac{1}{2\pi}\int_{2\pi} d\theta (1-\cos^2\theta)\exp(a\cos\theta) = I_0(a) - I_0''(a) = \frac{I_1(a)}{a} \quad \text{(C-57)}$$

and reference 9, 9.6.27, 9.6.26, and 10.2.13, and reference 6, 6.618 4.

Finally, the mean-square value of RV u is given by

$$E(u^2) = \frac{1}{M^2}\sum_{k,\ell=1}^{M} E\left\{\frac{(m+a_k)(m+a_\ell)}{\left[(m+a_k)^2+b_k^2\right]^{1/2}\left[(m+a_\ell)^2+b_\ell^2\right]^{1/2}}\right\} = \frac{1}{M}E\left\{\frac{(m+a)^2}{(m+a)^2+b^2}\right\} + \left(1-\frac{1}{M}\right)E^2\left\{\frac{m+a}{\left[(m+a)^2+b^2\right]^{1/2}}\right\}. \quad \text{(C-58)}$$

The first average in (C-58) may be expressed as

$$\frac{1}{M}\left[1 - E\left\{\frac{b^2}{(m+a)^2+b^2}\right\}\right] = \frac{1}{M}\left[1 - \frac{1-\exp(-m^2/2)}{m^2}\right], \quad \text{(C-59)}$$

upon using (C-55) and (C-56). The second average, by inspection of (C-51) and (C-52), is obviously m_u^2. Therefore the variance of RV u is

$$\sigma_u^2 = \frac{1}{M}\left[1 - \frac{1-\exp(-m^2/2)}{m^2} - m_u^2\right]. \quad \text{(C-60)}$$

It is immediately obvious from (C-56) and (C-60) that the variances of u and v are, in general, unequal. Thus for the second-order PDF, we have

$$p(u,v) \cong (2\pi\sigma_u\sigma_v)^{-1}\exp\left[-\frac{(u-m_u)^2}{2\sigma_u^2} - \frac{v^2}{2\sigma_v^2}\right]. \quad \text{(C-61)}$$

Then

$$P_D = \text{Prob}(\ell > T) = \text{Prob}\left(\sqrt{u^2+v^2} > T\right) = \iint_{u^2+v^2>T^2} du\,dv\,p(u,v)$$

$$\cong \iint_{u^2+v^2>T^2} du\,dv\,(2\pi\sigma_u\sigma_v)^{-1}\exp\left[-\frac{(u-m_u)^2}{2\sigma_u^2} - \frac{v^2}{2\sigma_v^2}\right] \equiv \tilde{P}_D. \quad \text{(C-62)}$$

Equation (C-62) gives the approximate detection probability $\tilde{P}_D$; its evaluation is in terms of integrals of elliptically bivariate Gaussian functions over offset circles, for which there are few tables available (e.g., see reference 10). Accordingly, a different approach is used here; i.e., the integral on v in (C-62) is first evaluated using (C-4). Then there follows

$$\tilde{P}_D = 1 - (2\pi)^{-1/2}\int_{-a}^{a} dx\,\exp\left[-\tfrac{1}{2}(x-b)^2\right]\left[2\Phi\left(\sqrt{c-dx^2}\right) - 1\right], \quad \text{(C-63)}$$

where

$$a = \frac{T}{\sigma_u} = \frac{\lambda}{\sqrt{2M}\,\sigma_u}\,, \quad b = \frac{m_u}{\sigma_u}\,, \quad c = \frac{T^2}{\sigma_v^2} = \frac{\lambda^2}{2M\sigma_v^2}\,, \quad d = \frac{\sigma_u^2}{\sigma_v^2}\,. \tag{C-64}$$

Equation (C-48) has been used in (C-64); the quantities required in (C-64) are given in (C-52), (C-56), and (C-60). Numerical evaluation is easily effected by using approximations to Φ (e.g., see reference 9, 26.2.17).

As the signal strength τ tends toward zero, there follows from (C-20), (C-52), (C-56), and (C-60),

$$m \to 0, \; m_u \to 0, \; \sigma_v^2 \to (2M)^{-1}, \; \sigma_u^2 \to (2M)^{-1}. \tag{C-65}$$

Therefore an approximation to P_F is afforded by (C-63):

$$\tilde{P}_F \equiv \tilde{P}_D(m=0) = \exp(-\lambda^2/2), \tag{C-66}$$

which is consistent with (C-50) for large M.

Equations (C-47) and (C-63) constitute the analytical results for Processor V. Values of λ are determined from (C-47), and then substituted in (C-63) and (C-64).

It is of interest to evaluate $\tilde{P}_D$ for small SNR, and to compare it with Processor II, which also uses amplitude samples. From (C-52), (C-56), and (C-60)

$$m_u \sim \left(\frac{\pi}{8}\right)^{1/2} m, \; \sigma_v^2 \sim \frac{1}{M}\left(\frac{1}{2} - \frac{1}{8}m^2\right), \; \sigma_u^2 \sim \frac{1}{M}\left(\frac{1}{2} - \frac{\pi-1}{8}m^2\right) \text{ as } m \to 0. \tag{C-67}$$

Therefore (C-62) yields

$$\begin{aligned} \tilde{P}_D &\sim \iint_{u^2+v^2>T^2} du\,dv\,(2\pi)^{-1}\,2M \exp\left[-M\left(u-\sqrt{\tfrac{\pi}{8}}\,m\right)^2 - M v^2\right] \\ &= Q\left(\frac{\sqrt{\pi}}{2}\sqrt{M}\,m,\, \sqrt{2M}\,T\right) = Q\left(\frac{\sqrt{\pi}}{2}\sqrt{M}\,m,\, \lambda\right) \text{ as } m \to 0, \end{aligned} \tag{C-68}$$

where we have changed to polar coordinates and used (C-8) and (C-48). A comparison of (C-68) with (C-9) reveals that Processor II functions equally as well as Processor V, while using a value for m that is $\sqrt{\pi}/2$ less. Thus, Processor V is

$$20 \log\left(\frac{2}{\sqrt{\pi}}\right) = 1.05 \text{ dB} \tag{C-69}$$

poorer than Processor II at low SNR and large M, where the central limit theorem is valid for the analysis of Processor V. Thus, loss of amplitude information does not cause much degradation in detectability; phase information is the major factor in determining detectability. It will be noticed that the amount

the performance of Processor V is below that of II is exactly the same as that by which Processor IV is below I. These are analogous processors, in that each pairing differs only in its use of amplitude information; they make the same use of phase information.

C.1.6 PROCESSOR VI; UNKNOWN SIGNAL PHASE; FITTED PHASE SAMPLES

The decision variable for this processor (table 1) was given as

$$\ell = \frac{1}{M}\sum_{K=1}^{M}\theta_K^2 - \left(\frac{1}{M}\sum_{K=1}^{M}\theta_K\right)^2 . \tag{C-70}$$

We are unable to exactly evaluate P_D or P_F in any simple form. Rather, we outline a numerical approach that could be used, and then derive approximations to the detection and false alarm probabilities.

<u>Exact Detection Probability</u>

The CF of RV ℓ in (C-70) is

$$f(\xi) = \int\cdots\int d\theta_1\cdots d\theta_M\, p(\theta_1)\cdots p(\theta_M)\exp\left[\frac{i\xi}{M}\sum_{K=1}^{M}\theta_K^2\right]\exp\left[-i\xi\left(\frac{1}{M}\sum_{K=1}^{M}\theta_K\right)^2\right]. \tag{C-71}$$

Now the second exponential in (C-71) can be simplified by means of the following artifice (reference 11, (22)):

$$\exp(-\sigma^2\mu^2/2) = \int dx\,(2\pi\sigma^2)^{-1/2}\exp\left(-\frac{x^2}{2\sigma^2} + i\mu x\right). \tag{C-72}$$

The quantity μ^2 on the left of (C-72) is replaced by μ on the right. Identifying

$$\mu = \sqrt{\xi}\,\frac{1}{M}\sum_{K=1}^{M}\theta_K\,, \quad \sigma = \sqrt{i2} = 1+i, \tag{C-73}$$

the multiple integral for the CF in (C-71) can then be expressed as

$$f(\xi) = \frac{(2\pi)^{-1/2}}{1+i}\int dx\,\exp(ix^2/4)\left\{\int d\theta\,p(\theta)\exp\left[\frac{i\xi}{M}\theta^2 + \frac{i\sqrt{\xi}}{M}x\theta\right]\right\}^M$$

$$= \frac{(2\pi\xi)^{-1/2}}{1+i}\int dx\,\exp\left(\frac{ix^2}{4\xi}\right)\left\{\int d\theta\,p(\theta)\exp\left[\frac{i\xi}{M}\theta^2 + \frac{i}{M}x\theta\right]\right\}^M . \tag{C-74}$$

For a particular value of ξ, either form of the inner integrals on θ in (C-74) can be quickly evaluated for many values of x by means of an FFT, permitting immediate evaluation of the outer integral on x. The PDF to use for $p(\theta)$ is given in (A-19) and depends on ψ_0. The final evaluation of the cumulative distribution of RV ℓ is available by using the methods in references 7 and 8. The exact P_F may be evaluated in a slightly simpler fashion since $p(\theta) = (2\pi)^{-1}$, $|\theta| < \pi$, in this case, and the integral on θ in (C-74) is the error function of a complex argument (reference 9, chapter 7).

Approximate Detection Probability

The RV ℓ in (C-70) is approximately Gaussian for large M. Therefore only its mean and variance must be computed to obtain approximations to P_D and P_F. Since P_D depends on the precise value of ψ_0, we will select the best value to maximize P_D. For the test in (C-70) where $|\theta_k| < \pi$, the best value of ψ_0 is zero. This point is discussed further in the main text.

In order to evaluate the mean and variance of RV ℓ in (C-70), we will need the moments of RV θ. For small SNR, we use the approximation given in (A-20) to order $P_0^2/\sigma^2 = m^2$; we obtain

$$\mu_n \equiv E(\theta^n) = \int d\theta\, \theta^n\, p(\theta)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \theta^n \left[1 + \sqrt{\frac{\pi}{2}}\, m\cos\theta + \frac{1}{2} m^2 \cos(2\theta)\right], \text{ to order } m^2. \tag{C-75}$$

It follows that $\mu_n = 0$ for n odd, which is the result of our choice of $\psi_0 = 0$. Using integration by parts for n even, we obtain

$$\mu_0 = 1,$$
$$\mu_2 = \frac{\pi^2}{3} - \sqrt{2\pi}\, m + \frac{1}{4} m^2, \text{ to order } m^2, \tag{C-76}$$
$$\mu_4 = \frac{\pi^4}{5} - 2\sqrt{2\pi}\,(\pi^2 - 6)\, m + \frac{1}{4}(2\pi^2 - 3)\, m^2, \text{ to order } m^2.$$

The mean of RV ℓ is obtained from (C-70) as follows:

$$E\{\ell\} = \frac{1}{M} M\mu_2 - \frac{1}{M^2}\left[M\mu_2 - (M^2 - M)\mu_1^2\right] = \mu_2\left(1 - \frac{1}{M}\right) \equiv m_\ell . \tag{C-77}$$

The square of RV ℓ can be expressed as

$$\ell^2 = \frac{1}{M^2}\sum_{k,m=1}^{M} \theta_k^2 \theta_m^2 - \frac{2}{M^3}\sum_{k,m,n=1}^{M} \theta_k^2 \theta_m \theta_n + \frac{1}{M^4}\sum_{k,m,n,p=1}^{M} \theta_k \theta_m \theta_n \theta_p . \tag{C-78}$$

The means of each of the three summations in (C-78) are, respectively,

$$M\mu_4 + (M^2 - M)\mu_2^2$$

$$M\mu_4 + 2M(M-1)\mu_3\mu_1 + M(M-1)\mu_2^2 + M(M-1)(M-2)\mu_2\mu_1^2, \tag{C-79}$$

$$M\mu_4 + 4M(M-1)\mu_3\mu_1 + 3M(M-1)\mu_2^2 + 6M(M-1)(M-2)\mu_2\mu_1^2 + M(M-1)(M-2)(M-3)\mu_1^4 .$$

The mean-square value is then available from (C-78) and (C-79), and the variance of ℓ is, finally,

$$\mathrm{Var}\{\ell\} = E\{\ell^2\} - E^2\{\ell\} = \frac{1}{M}\left(1-\frac{1}{M}\right)^2\left[\mu_4 - \mu_2^2\,\frac{M-3}{M-1}\right] \equiv \sigma_\ell^2 . \tag{C-80}$$

The ratio

$$\frac{m_\ell}{\sigma_\ell} = \sqrt{M}\,\frac{\mu_2}{\left[\mu_4 - \mu_2^2\,\frac{M-3}{M-1}\right]^{1/2}} \tag{C-81A}$$

$$\cong \sqrt{M}\,\frac{E\{\theta^2\}}{\text{Std. Dev.}\{\theta^2\}} \quad \text{for large } M. \tag{C-81B}$$

The last relation would be obtained from (C-70) by simply dropping the last term. Thus an approximation to Processor VI would be afforded by $\frac{1}{M}\sum_{K=1}^{M}\theta_K^2$; this possibility has not been pursued further, however.

The probability of RV ℓ <u>not</u> exceeding a threshold T is approximated according to

$$\mathrm{Prob}(\ell < T) \cong \int_{-\infty}^{T} d\ell\,(2\pi\sigma_\ell^2)^{-1/2}\exp\left[-\frac{(\ell-m_\ell)^2}{2\sigma_\ell^2}\right] = \Phi\left(\frac{T-m_\ell}{\sigma_\ell}\right), \tag{C-82}$$

by using (C-4). The quantity m_ℓ/σ_ℓ is given by (C-81A), (C-77), and (C-80) in terms of μ_2 and μ_4, which, in turn, are given by (C-76), or, more generally, by (C-75). For large M (and $\psi_0 = 0$), we obtain the approximations

$$\tilde{P}_F = \Phi(-\lambda),\ \tilde{P}_D = \Phi\left(\frac{3}{\pi}\sqrt{\frac{5}{2\pi}}\sqrt{M}\,m - \lambda\right) = \Phi\left(.852\sqrt{M}\,m - \lambda\right), \text{ to order } m, \tag{C-83}$$

where $\lambda \equiv \frac{\sqrt{5M}}{2}\left(1-\frac{3}{\pi}T\right)$. A comparison of (C-83) with (C-5) shows that, when $\psi_0 = 0$, Processor VI is

$$20 \log(1/.852) = 1.39 \text{ dB} \tag{C-84}$$

poorer than the optimum phase-coherent processor (I) for large M and small SNR, whereas a comparison of (C-83) with (C-40) et seq. shows that Processor VI is 0.34 dB (1.39 - 1.05) poorer than Processor IV for large M and small SNR. Thus, for known signal phase, ψ_0, the nonoptimum combination of phase samples, (C-70), is only 0.34 dB poorer than the optimum combination (see table 1) for large M and small SNR. However, these comparisons may not be relevant for the range of useful P_F and P_D; this relevance can be determined only from the detailed receiver operating characteristics presented in the main text.

C.1.7 PROCESSOR VII: UNKNOWN SIGNAL PHASE; PHASE-DIFFERENCE SAMPLES

The decision variable for this processor (table 1) was given as

$$\mathrm{Re}\left\{\sum_{K=2}^{M} \exp(i\theta_K - i\theta_{K-1})\right\}. \tag{C-85}$$

We actually consider a more general processor here, namely, one that uses weighted phase differences

$$\ell = \mathrm{Re}\left\{\sum_{K=2}^{M} R_K^{\mu} R_{K-1}^{\mu} \exp(i\theta_K - i\theta_{K-1})\right\}, \tag{C-86}$$

because better performance may be attainable for $\mu \neq 0$ than for $\mu = 0$. In trade, however, the CFAR capability is lost for $\mu \neq 0$.

Exact Detection Probability

The one case in which a closed-form expression for the CF of RV ℓ in (C-86) is attainable is where $\mu = 1$. To show this, we use (5) and (C-15) to express

$$\ell = \sigma^2 \sum_{K=2}^{M} \left[(m + a_K)(m + a_{K-1}) + b_K b_{K-1}\right] \equiv \ell_a + \ell_b. \tag{C-87}$$

The CF of ℓ is given by the product of the CFs for RVs ℓ_a and ℓ_b since $\{a_K\}$ and $\{b_K\}$ are independent (see (C-18)). Also, the CF for ℓ_b is a special case of that for ℓ_a, obtained by setting $m = 0$ in the quantity ℓ_a. In order to obtain the CF for RV ℓ_a, we use reference 1, appendix C: Matrix $\underset{\sim}{B}$ has elements

$$B_{ij} = \begin{cases} \sigma^2/2, & |i-j| = 1 \\ 0, & \text{otherwise} \end{cases}, \quad 1 \le i,j \le M, \tag{C-88}$$

and all the elements of column matrix $\underset{\sim}{m}$ are equal to m. The covariance of RVs $\{a_k\}$ is equal to the identity matrix; therefore matrix $\underset{\approx}{A}$ is equal to $\underset{\sim}{B}$ (reference 1, (C-8)). Let the normalized modal matrix and characteristic value matrix of $\underset{\sim}{B}$ be denoted by $\underset{\sim}{Q}$ and $\underset{\sim}{\lambda}$, respectively. Then the CF of ℓ_a is given by

$$\prod_{k=1}^{M}\left\{(1-i2\lambda_k\xi)^{-1/2}\exp\left(\frac{i\mu_k^2\lambda_k\xi}{1-i2\lambda_k\xi}\right)\right\}, \tag{C-89}$$

where matrix

$$\underset{\sim}{\mu} = \underset{\sim}{Q}^T\underset{\sim}{m} = m\underset{\sim}{Q}^T\underset{\sim}{1}. \tag{C-90}$$

It follows that the CF of RV ℓ_b is given by

$$\prod_{k=1}^{M}\left\{(1-i2\lambda_k\xi)^{-1/2}\right\} \tag{C-91}$$

and, therefore, the CF of RV ℓ in (C-87) is

$$f(\xi) = \prod_{k=1}^{M}\left\{(1-i2\lambda_k\xi)^{-1}\exp\left(\frac{i\mu_k^2\lambda_k\xi}{1-i2\lambda_k\xi}\right)\right\}. \tag{C-92}$$

Evaluation of the characteristic values $\{\lambda_k\}$ of $\underset{\sim}{B}$ is possible from reference 12, (67) and (68), and are given by

$$\lambda_k = \sigma^2\cos\left(\frac{k\pi}{M+1}\right),\ 1 \le k \le M. \tag{C-93}$$

Evaluation of the normalized modal matrix $\underset{\sim}{Q}$ of matrix $\underset{\sim}{B}$ in (C-88) requires computer computation, and (C-92) can be evaluated only with such aid. The cumulative probability distribution and, hence, the operating characteristics of Processor VII for $\mu=1$ in (C-86) are then available by means of the techniques described in references 7 and 8. This exact approach to the detection probability for $\mu=1$ has not been pursued further in this report.

Exact False Alarm Probabilities

P_F for $\mu=1$. The false alarm probability for $\mu=1$ is immediately available from the above results. For $m=0$, (C-90) yields $\mu_k=0$, and (C-92) becomes

$$f(\xi) = \prod_{k=1}^{M}(1-i2\lambda_k\xi)^{-1}. \tag{C-94}$$

The PDF of RV ℓ in (C-87) is then given by

$$p(\ell) = \frac{1}{2}\sum_{m=1}^{[M/2]}\frac{C_m}{\lambda_m}\exp\left(-\frac{|\ell|}{2\lambda_m}\right), \tag{C-95}$$

where

$$C_m = \prod_{\substack{k=1 \\ k \neq m}}^{M} \frac{\lambda_m}{\lambda_m - \lambda_k}, \quad 1 \leq m \leq [M/2], \tag{C-96}$$

and $\{\lambda_k\}$ are given in (C-93). The probability of false alarm is

$$\begin{aligned} P_F(\mu=1) &= \text{Prob}(\ell > T) = \int_T^{\infty} d\ell \; p(\ell) \\ &= \sum_{m=1}^{[M/2]} C_m \exp\left(-\frac{T}{2\lambda_m}\right) \quad \text{for } T \geq 0. \end{aligned} \tag{C-97}$$

It is convenient to define a normalized threshold as

$$\lambda = \frac{T}{\sigma^2 \sqrt{2(M-1)}}, \tag{C-98}$$

in which case (C-97) becomes

$$P_F(\mu=1) = \sum_{m=1}^{[M/2]} C_m \exp\left(-\frac{\sqrt{(M-1)/2}\;\lambda}{\cos\left(\frac{m\pi}{M+1}\right)}\right), \quad \lambda \geq 0 \tag{C-99}$$

The reason for the scaling in (C-98) is that P_F in (C-99) now approaches $\Phi(-\lambda)$ as $M \to \infty$. To see this, we can either use (C-87) and note that ℓ approaches a Gaussian RV with a zero mean and a variance of $\sigma^4 2(M-1)$ for large M, or we can investigate (C-94) for large M: We have

$$f(\xi) \cong \frac{1}{1 - i2\xi \sum_{k=1}^{M} \lambda_k + (-i2\xi)^2 \frac{1}{2} \sum_{k \neq \ell}^{M} \lambda_k \lambda_\ell} \quad \text{for small } \xi. \tag{C-100}$$

But the sum of characteristic values is zero because the diagonal elements of $\underset{\sim}{B}$ in (C-88) are all zero. Also

$$\begin{aligned} \sum_{k \neq \ell}^{M} \lambda_k \lambda_\ell &= \sum_{k,\ell=1}^{M} \lambda_k \lambda_\ell - \sum_{k=1}^{M} \lambda_k^2 = \left(\sum_{k=1}^{M} \lambda_k\right)^2 - \sum_{k=1}^{M} \lambda_k^2 \\ &= -\sigma^4 \sum_{k=1}^{M} \cos^2\left(\frac{k\pi}{M+1}\right) = -\sigma^4 \frac{M-1}{2}, \end{aligned} \tag{C-101}$$

using (C-93). Therefore

$$f(\xi) \cong \frac{1}{1 + \sigma^4 (M-1)\xi^2} \cong \exp\left[-\sigma^4 (M-1)\xi^2\right] \quad \text{for small } \xi, \tag{C-102}$$

which is a Gaussian CF with a zero mean and a variance of $2\sigma^4(M-1)$. Therefore the choice of $\lambda(P_F, M)$ in the exact relation (C-99) becomes relatively independent of M, for large M. Table C-3 gives values of λ, as determined from (C-99), for $M=25$ and $\mu=1$.

Table C-3. Thresholds Required for Processor VII for M = 25 and μ = 1; Known Signal Frequency

P_F (μ = 1)	λ (P_F, M = 25)
10^{-1}	1.25712
10^{-2}	2.40560
10^{-3}	3.35438
10^{-4}	4.21662

P_F for $\mu=0$. The false alarm probability for $\mu=0$ in (C-86) can also be derived in a tractable form. This processor is the phase-difference processor, (C-85), that we express as

$$\ell = \sum_{K=2}^{M} \left(\cos\theta_K \cos\theta_{K-1} + \sin\theta_K \sin\theta_{K-1}\right). \tag{C-103}$$

The CF of ℓ is then

$$f(\xi) = E\left\{\exp\left[i\xi\sum_{K=2}^{M}\left(\cos\theta_K\cos\theta_{K-1} + \sin\theta_K\sin\theta_{K-1}\right)\right]\right\}. \tag{C-104}$$

Now the average on θ_1 in (C-104) is (see (C-15) through (C-18))

$$\frac{1}{2\pi}\int_{2\pi} d\theta_1 \exp\left[i\xi\left(\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1\right)\right] = J_0(\xi). \tag{C-105}$$

A similar procedure applied, in turn, to $\theta_2, \theta_3, \ldots, \theta_{M-1}$ yields the closed-form CF for ℓ:

$$f(\xi) = J_0^{M-1}(\xi). \tag{C-106}$$

Then, according to reference 7, (7), the exact false alarm probability is

$$\begin{aligned} P_F(\mu=0) = \operatorname{Prob}(\ell > T) &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{d\xi}{\xi} J_0^{M-1}(\xi)\sin(\xi T) \\ &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty dx\, \frac{\sin x}{x} J_0^{M-1}\left(\sqrt{\frac{2}{M-1}}\,\frac{x}{\lambda}\right), \end{aligned} \tag{C-107}$$

where normalized threshold

$$\lambda \equiv \sqrt{\frac{2}{M-1}}\,T. \tag{C-108}$$

By an argument similar to that in (C-28) through (C-32), we find

$$P_F(\mu=0) \sim \Phi(-\lambda) \quad \text{as } M \to \infty. \tag{C-109}$$

Table C-4 gives values of λ, as determined from (C-107), for $M=25$ and $\mu=0$.

Table C-4. Thresholds Required for Processor VII for M = 25 and μ = 0; Known Signal Frequency

P_F (μ = 0)	λ (P_F, M = 25)
10^{-1}	1.28615
10^{-2}	2.31149
10^{-3}	3.03591
10^{-4}	3.60941

Approximate Detection Probability

The RV ℓ in (C-86) for large M is approximately Gaussian. Accordingly, we need evaluate only its mean and variance. By using (C-15) and (C-20), we first express (C-86) as

$$\ell = \sigma^{2\mu} \sum_{K=2}^{M} \frac{(m+a_K)(m+a_{K-1}) + b_K b_{K-1}}{\left[(m+a_K)^2 + b_K^2\right]^{\nu} \left[(m+a_{K-1})^2 + b_{K-1}^2\right]^{\nu}}, \tag{C-110}$$

where

$$\nu \equiv \frac{1-\mu}{2}. \tag{C-111}$$

Then we define

$$C_K \equiv \frac{m+a_K}{\left[(m+a_K)^2 + b_K^2\right]^{\nu}}, \quad S_K \equiv \frac{b_K}{\left[(m+a_K)^2 + b_K^2\right]^{\nu}}, \tag{C-112}$$

and express

$$\ell = \sigma^{2\mu}(M-1)u, \tag{C-113}$$

where

$$u \equiv \frac{1}{M-1} \sum_{K=2}^{M} \left(C_K C_{K-1} + S_K S_{K-1}\right). \tag{C-114}$$

At this point, we derive several generic results for the RVs $\{C_k\}$ and $\{S_k\}$ that we will need both now and later. (The derivation of the mean and variance of u resumes at equation (C-121).) We shall use (C-18) et seq. freely. We have the mean

$$\overline{C_k} = \iint da\, db\, \frac{m+a}{\left[(m+a)^2+b^2\right]^{\nu}}\, \frac{1}{2\pi} \exp\left(-\frac{a^2+b^2}{2}\right)$$

$$= \int_0^\infty dr\, r\, \frac{1}{2\pi} \int_{2\pi} d\theta\, r^{1-2\nu} \cos\theta \exp\left[-\frac{r^2-2mr\cos\theta+m^2}{2}\right]$$

$$= \int_0^\infty dr\, r^{2-2\nu} \exp\left(-\frac{r^2+m^2}{2}\right) I_1(mr) \qquad \text{(C-115)}$$

$$= 2^{-\nu}\, \Gamma(2-\nu)\, m \exp(-m^2/2)\, {}_1F_1(2-\nu; 2; m^2/2)$$

$$= 2^{\frac{\mu-1}{2}}\, \Gamma\left(\frac{3+\mu}{2}\right) m \exp(-m^2/2)\, {}_1F_1\left(\frac{3+\mu}{2}; 2; \frac{m^2}{2}\right) \equiv \overline{C},$$

where we have transformed to polar coordinates according to $a = r\cos\theta - m$, $b = r\sin\theta$, and used reference 6, 8.431 5, 6.631 1, and (C-111). Also,

$$\overline{S_k} = \int_0^\infty dr\, r\, \frac{1}{2\pi} \int_{2\pi} d\theta\, r^{1-2\nu} \sin\theta \exp\left[-\frac{r^2-2mr\cos\theta+m^2}{2}\right] = 0, \qquad \text{(C-116)}$$

using the oddness of the θ-integrand.

The cross-moment

$$\overline{C_k S_k} = \int_0^\infty dr\, r\, \frac{1}{2\pi} \int_{2\pi} d\theta\, r^{2-4\nu} \cos\theta \sin\theta \exp\left[-\frac{r^2-2mr\cos\theta+m^2}{2}\right] = 0, \qquad \text{(C-117)}$$

using the oddness of the θ-integrand. Also,

$$\overline{S_k^2} = \int_0^\infty dr\, r\, \frac{1}{2\pi} \int_{2\pi} d\theta\, r^{2-4\nu} \sin^2\theta \exp\left[-\frac{r^2-2mr\cos\theta+m^2}{2}\right]$$

$$= \int_0^\infty dr\, r^{3-4\nu} \exp\left(-\frac{r^2+m^2}{2}\right) \frac{I_1(mr)}{mr} \qquad \text{(C-118)}$$

$$= 4^{-\nu}\, \Gamma(2-2\nu) \exp(-m^2/2)\, {}_1F_1(2-2\nu; 2; m^2/2)$$

$$= 2^{\mu-1}\, \Gamma(1+\mu) \exp(-m^2/2)\, {}_1F_1(1+\mu; 2; m^2/2) \equiv \overline{S^2},$$

using reference 6, 8.431 3 and 6.631 1, and (C-111). Finally,

$$\overline{C_K^2} = \int_0^\infty dr\, r\, \frac{1}{2\pi}\int_{2\pi} d\theta\, r^{2-4\nu} \cos^2\theta \exp\left[-\frac{r^2-2mr\cos\theta+m^2}{2}\right]. \qquad \text{(C-119)}$$

We express $\cos^2\theta = 1-\sin^2\theta$ in (C-119) and use reference 6, 6.631 1, and (C-118) and (C-111) to obtain

$$\overline{C_K^2} = \int_0^\infty dr\, r^{3-4\nu} \exp\left(-\frac{r^2+m^2}{2}\right) I_0(mr) - \sigma_s^2$$

$$= 2^{\mu-1}\,\Gamma(1+\mu)\exp(-m^2/2)\left[2\,{}_1F_1(1+\mu;1;m^2/2) - {}_1F_1(1+\mu;2;m^2/2)\right] \equiv \overline{C^2}. \qquad \text{(C-120)}$$

Now, we resume the calculation of the statistics of RV u in (C-114). We have

$$\overline{u} = \overline{C}^2 + \overline{S}^2 = \overline{C}^2 \equiv m_u, \qquad \text{(C-121)}$$

using (C-115); the second moment

$$\overline{u^2} = \frac{1}{(M-1)^2}\sum_{K,\ell=2}^{M} E\left\{(C_K C_{K-1} + S_K S_{K-1})(C_\ell C_{\ell-1} + S_\ell S_{\ell-1})\right\}$$

$$= \frac{1}{(M-1)^2}\Bigg[\sum_{K=2}^{M} E\,(C_K C_{K-1} + S_K S_{K-1})^2$$

$$+ \sum_{K=2}^{M-1} E\left\{(C_K C_{K-1} + S_K S_{K-1})(C_{K+1} C_K + S_{K+1} S_K)\right\} \qquad \text{(C-122)}$$

$$+ \sum_{K=3}^{M} E\left\{(C_K C_{K-1} + S_K S_{K-1})(C_{K-1} C_{K-2} + S_{K-1} S_{K-2})\right\}$$

$$+ \sum_{\substack{K,\ell=2 \\ |K-\ell|\geq 2}}^{M} E\left\{(C_K C_{K-1} + S_K S_{K-1})(C_\ell C_{\ell-1} + S_\ell S_{\ell-1})\right\}\Bigg].$$

The four averages in (C-122) are given, respectively, by

$$\overline{C_K^2}\;\overline{C_{K-1}^2} + 2\,\overline{S_K C_K}\;\overline{S_{K-1} C_{K-1}} + \overline{S_K^2}\;\overline{S_{K-1}^2} = \overline{C^2}^2 + \overline{S^2}^2,$$

$$\overline{C_K^2}\;\overline{C_{K-1}}\;\overline{C_{K+1}} + \overline{C_K S_K}\;\overline{C_{K-1}}\;\overline{S_{K+1}} + \overline{C_K S_K}\;\overline{S_{K-1}}\;\overline{C_{K+1}} + \overline{S_K^2}\;\overline{S_{K-1}}\;\overline{S_{K+1}} = \overline{C^2}\;\overline{C}^2, \qquad \text{(C-123)}$$

$$\overline{C_K}\;\overline{C_{K-1}^2}\;\overline{C_{K-2}} + \overline{C_K}\;\overline{C_{K-1} S_{K-1}}\;\overline{S_{K-2}} + \overline{S_K}\;\overline{S_{K-1} C_{K-1}}\;\overline{C_{K-2}} + \overline{S_K}\;\overline{S_{K-1}^2}\;\overline{S_{K-2}} = \overline{C^2}\;\overline{C}^2,$$

$$\overline{C}^4,$$

upon using (C-115) through (C-120). Combining (C-121) through (C-123), it follows for the variance of u that

$$\sigma_u^2 = \frac{1}{M-1}\left[\left(\overline{C^2} + \overline{C}^2\right)^2 + \overline{S}^2{}^2 - 4\overline{C}^4 - \frac{2}{M-1}\overline{C}^2\left(\overline{C^2} - \overline{C}^2\right)\right]. \qquad \text{(C-124)}$$

The PDF of u is then approximately

$$p(u) \cong \frac{1}{\sqrt{2\pi}\,\sigma_u} \exp\left[-\frac{(u-m_u)^2}{2\sigma_u^2}\right], \qquad \text{(C-125)}$$

and

$$P_D = \text{Prob}(\ell > T) = \text{Prob}\left(u > \frac{T}{\sigma^{2\mu}(M-1)}\right)$$

$$\cong \Phi\left(\frac{m_u}{\sigma_u} - \frac{T}{\sigma_u \sigma^{2\mu}(M-1)}\right) \equiv \tilde{P}_D . \qquad \text{(C-126)}$$

The quantities m_u and σ_u are available from (C-121) and (C-124), upon employment of (C-115), (C-118), and (C-120). The approximate false alarm probability is obtained by setting $m=0$ in the above results and is

$$\tilde{P}_F = \Phi(-\lambda), \qquad \text{(C-127)}$$

where normalized threshold

$$\lambda \equiv \frac{T}{\sqrt{2(M-1)}\, 2^{\mu-1}\Gamma(1+\mu)\sigma^{2\mu}} . \qquad \text{(C-128)}$$

(This agrees with (C-98) and (C-108) for $\mu=1$ and $\mu=0$, respectively.) Then (C-126) becomes

$$\tilde{P}_D = \Phi\left(\frac{m_u}{\sigma_u} - \frac{2^{\mu-\frac{1}{2}}\Gamma(1+\mu)}{\sigma_u (M-1)^{1/2}}\lambda\right). \qquad \text{(C-129)}$$

Equations (C-127) and (C-129) constitute the approximations to P_F and P_D for arbitrary μ. Threshold λ is determined from (C-127) for a specified $\tilde{P}_F$, and then employed in (C-129) to evaluate $\tilde{P}_D$. However, for $\mu=0$ or $\mu=1$, λ should be determined from (C-107) or (C-99), respectively, and substituted in (C-129) for evaluation of $\tilde{P}_D$. (Of course, the best approach for $\mu=1$ is to evaluate the exact P_D via (C-92).) The quantities m_u and σ_u are available in (C-121) and (C-124).

Approximations for Small m

For small SNRs per sample, $m \ll 1$. In this case, the following approximations result, to order m:

$$
\begin{aligned}
\overline{C} &\sim 2^{\frac{\mu-1}{2}}\,\Gamma\!\left(\frac{3+\mu}{2}\right) m,\\
\overline{S^2} &\sim 2^{\mu-1}\,\Gamma(1+\mu),\\
\overline{C^2} &\sim 2^{\mu-1}\,\Gamma(1+\mu),\\
m_u &\sim 2^{\mu-1}\,\Gamma^2\!\left(\frac{3+\mu}{2}\right) m^2,\\
\sigma_u &\sim \left(\frac{2}{M-1}\right)^{1/2} 2^{\mu-1}\,\Gamma(1+\mu).
\end{aligned}
\qquad \text{(C-130)}
$$

Then (C-129) becomes

$$
\tilde{P}_D \sim \Phi\left(\frac{\Gamma^2\left(\frac{3+\mu}{2}\right)}{\Gamma(1+\mu)} \left(\frac{M-1}{2}\right)^{1/2} m^2 - \lambda\right) \text{ as } m \to 0. \qquad \text{(C-131)}
$$

Two important features should be noted from (C-131). First, the dependence on m is according to m^2, not according to m as for Processor I, (C-5); Processor IV, (C-40); and Processor VI, (C-83). This result is also discernible in the exact CF calculation (C-92) for $\mu = 1$, where the dependence of μ_K^2 is according to m^2 (see (C-90)). Thus the phase-difference processors have a small-signal suppression effect for all values of weighting constant μ in (C-86). Second, there is an optimum value of μ to maximize the coefficient of m^2 in (C-131). The calculations shown in table C-5 indicate that the best value of weighting μ to use in (C-86) is unity; however, the factor has a broad maximum about the point $\mu = 1$. (Also, the best value of μ for larger SNRs may not be 1.) The loss in detectability at $\mu = 0$ versus $\mu = 1$ is available from (C-131) and table C-5 and is 10 log $(4/\pi)$ = 1.05 dB for small SNR; once again, the loss of amplitude information incurs a 1.05-dB degradation in performance for small SNR and large M.

Table C-5. Multiplying Factor Used in (C-131)

μ	0	1/2	1	1-1/2	2
$\Gamma^2\left(\frac{3+\mu}{2}\right)/\Gamma(1+\mu)$	$\pi/4$ .785	.953	1	.966	.884

C. 2 UNKNOWN SIGNAL FREQUENCY

In accordance with the discussion at the end of section 2, the only processor we will consider here is:

C. 2. 7 PROCESSOR VII; UNKNOWN SIGNAL PHASE; PHASE-DIFFERENCE SAMPLES

The decision variable for this processor (table 2) was given by

$$\ell = \left| \sum_{k=2}^{M} \exp\left(i\theta_k - i\theta_{k-1}\right) \right| . \tag{C-132}$$

We actually consider a more general processor here, namely, weighted phase-differences

$$\ell = \left| \sum_{k=2}^{M} R_k^{\mu} R_{k-1}^{\mu} \exp\left(i\theta_k - i\theta_{k-1}\right) \right| , \tag{C-133}$$

because better performance may be attainable for $\mu \neq 0$ than for $\mu = 0$. An exact relation for the detection probability is not available for any value of μ. (For $\mu = 1$, the fundamental problem is evaluation of the PDF of $\left| \sum_{k=2}^{M} z_k z_{k-1}^* \right|$, where $\{z_k\}$ are complex independent Gaussian RVs with nonzero means.) We will, however, obtain approximations to the detection and false alarm probabilities for general values of μ.

<u>Exact False Alarm Probabilities</u>

In a manner similar to that given in (C-110) through (C-114), we express (C-133) as

$$\ell = \sigma^{-2\mu}(M-1)\left|u + iv\right| \equiv \sigma^{-2\mu}\left|\tilde{u} + i\tilde{v}\right| , \tag{C-134}$$

where

$$u \equiv \frac{1}{M-1} \sum_{k=2}^{M} \left(C_k C_{k-1} + S_k S_{k-1}\right),$$

$$v = \frac{1}{M-1} \sum_{k=2}^{M} \left(S_k C_{k-1} - C_k S_{k-1}\right). \tag{C-135}$$

The second-order CF of RVs u and v is

$$f(\xi, \eta) = E\left\{\exp\left(i\xi u + i\eta v\right)\right\}$$

$$= E\left\{\exp\left[\frac{i\xi}{M-1}\sum_{k=2}^{M}\left(C_k C_{k-1} + S_k S_{k-1}\right) + \frac{i\eta}{M-1}\sum_{k=2}^{M}\left(S_k C_{k-1} - C_k S_{k-1}\right)\right]\right\} . \tag{C-136}$$

Now if we define

$$\alpha = (\xi C_2 + \eta S_2)/(M-1)\,, \quad \beta = (\xi S_2 - \eta C_2)/(M-1), \tag{C-137}$$

the average on C_1 and S_1 in (C-136) becomes

$$\begin{aligned}
& E\left\{\exp\left(i\alpha C_1 + i\beta S_1\right)\right\} \\
&= \iint da\, db\, \exp\left[i\, \frac{\alpha a + \beta b}{(a^2+b^2)^{\mu}}\right] \frac{1}{2\pi} \exp\left(-\frac{a^2+b^2}{2}\right) \\
&= \int_0^{\infty} dr\, r \int_{2\pi} d\theta\, \exp\left[i r^{\mu}(\alpha\cos\theta + \beta\sin\theta)\right] \frac{1}{2\pi} \exp(-r^2/2) \\
&= \int_0^{\infty} dr\; r\; \exp(-r^2/2)\, J_0\left(\sqrt{\alpha^2+\beta^2}\; r^{\mu}\right) \\
&= \int_0^{\infty} dr\, r\, \exp(-r^2/2)\, J_0\left(\frac{1}{M-1}\sqrt{\xi^2+\eta^2}\sqrt{C_2^2+S_2^2}\; r^{\mu}\right),
\end{aligned} \tag{C-138}$$

where we have employed (C-111) and (C-112), set $m=0$, and have changed to polar coordinates. At this point, we are stymied in attempting to evaluate the average on C_2 and S_2 in (C-136), except in the two special cases of $\mu=0$ and $\mu=1$.

<u>P_F for $\mu=0$.</u> For $\mu=0$, we observe from (C-111) and (C-112) that $C_2^2+S_2^2=1$, in which case (C-138) becomes

$$J_0\left(\frac{1}{M-1}\sqrt{\xi^2+\eta^2}\right). \tag{C-139}$$

A similar step-by-step procedure for the remaining averages in (C-136) yields

$$f(\xi,\eta) = J_0^{M-1}\left(\frac{1}{M-1}\sqrt{\xi^2+\eta^2}\right). \tag{C-140}$$

Then by the procedure given in (C-44) through (C-46), we obtain

$$\begin{aligned}
\mathrm{Prob}(\ell > T) = P_F(\mu=0) &= 1 - \int_0^{\infty} dx\, J_1(x)\, J_0^{M-1}(x/T) \\
&= 1 - \int_0^{\infty} dx\, J_1(x)\, J_0^{M-1}\left(\sqrt{\frac{2}{M-1}}\,\frac{x}{\lambda}\right),
\end{aligned} \tag{C-141}$$

where normalized threshold

$$\lambda \equiv \sqrt{\frac{2}{M-1}}\, T. \tag{C-142}$$

The scaling in (C-142) is chosen such that

$$P_F(\mu = 0) \sim \exp(-\lambda^2/2) \quad \text{as} \quad M \to \infty, \tag{C-143}$$

which follows when (C-47) through (C-50) are used. Table C-6 gives values of λ, as determined from (C-141), for $M = 25$ and $\mu = 0$.

Table C-6. Thresholds Required for Processor VII for M = 25 and $\mu = 0$; Unknown Signal Frequency

$P_F\ (\mu = 0)$	$\lambda\ (P_F,\ M = 25)$
10^{-1}	2.14258
10^{-2}	2.99250
10^{-3}	3.61779
10^{-4}	4.12110

$\underline{P_F \text{ for } \mu = 1}$. For $\mu = 1$, we observe from (C-111) and (C-112) that $C_K = a_K$ and $S_K = b_K$. Then (C-134) and (C-135) become, respectively,

$$\begin{aligned} \tilde{u} &= \sum_{K=2}^{M} \left(a_K a_{K-1} + b_K b_{K-1} \right), \\ \tilde{v} &= \sum_{K=2}^{M} \left(b_K a_{K-1} - a_K b_{K-1} \right). \end{aligned} \tag{C-144}$$

The second-order CF of $\tilde{u}$ and $\tilde{v}$ is

$$E\left\{ \exp\left[i\xi \sum_{K=2}^{M} \left(a_K a_{K-1} + b_K b_{K-1} \right) + i\eta \sum_{K=2}^{M} \left(b_K a_{K-1} - a_K b_{K-1} \right) \right] \right\}. \tag{C-145}$$

The average on a_1 in (C-145) is

$$\int da_1 \exp\left[i\left(\xi a_2 + \eta b_2 \right) a_1 \right] (2\pi)^{-1/2} \exp(-a_1^2/2) = \exp\left[-\tfrac{1}{2}\left(\xi a_2 + \eta b_2 \right)^2 \right], \tag{C-146}$$

while the average on b_1 yields

$$\exp\left[-\tfrac{1}{2}(\xi b_2 - \eta a_2)^2\right]. \tag{C-147}$$

The product of these averages is

$$\exp\left[-\tfrac{1}{2}(\xi^2+\eta^2)(a_2^2+b_2^2)\right]. \tag{C-148}$$

(This result is also obtainable from (C-138) for $\mu=1$, using reference 6, 6.631 4, when we observe the scale factor $M-1$ used in (C-134).) Now the averages on a_2 and b_2 in (C-145) are very similar to those in (C-146) and (C-147), except for the added exponential (C-148). Generally, the average on a_K takes the form

$$\frac{1}{\sqrt{2\pi}}\int da_K \exp\left[i(\xi a_{K+1}+\eta b_{K+1})a_K\right]\exp\left[-\frac{a_K^2}{2}\left\{1+(\xi^2+\eta^2)\frac{T_{K-2}(\xi^2+\eta^2)}{T_{K-1}(\xi^2+\eta^2)}\right\}\right]\left[T_{K-1}(\xi^2+\eta^2)\right]^{-1/2}$$

$$= \frac{1}{\sqrt{T_K(\xi^2+\eta^2)}}\exp\left[-\tfrac{1}{2}(\xi a_{K+1}+\eta b_{K+1})^2\,\frac{T_{K-1}(\xi^2+\eta^2)}{T_K(\xi^2+\eta^2)}\right],\quad 1\le K\le M-1, \tag{C-149}$$

where $T_K(y)$ is a polynomial in y that satisfies the recurrence relation

$$T_K(y) = T_{K-1}(y) + y\,T_{K-2}(y),\ K\ge 1;\ T_{-1}(y)=0,\ T_0(y)=1. \tag{C-150}$$

In fact,

$$T_M(y) = \sum_{K=0}^{[M/2]}\binom{M-K}{K}y^K = {}_2F_1\left(\frac{1-M}{2}, \frac{-M}{2}; -M; -4y\right). \tag{C-151}$$

The first few polynomials are*

$$\begin{aligned} T_0(y) &= 1\\ T_1(y) &= 1\\ T_2(y) &= 1+y\\ T_3(y) &= 1+2y\\ T_4(y) &= 1+3y+y^2\\ T_5(y) &= 1+4y+3y^2. \end{aligned} \tag{C-152}$$

The product of the averages on a_K and b_K in (C-145) is then given by

---

*The factorization of $T_M(y)$ is presented in (C-162) and (C-161).

$$\frac{1}{T_K(\xi^2+\eta^2)} \exp\left[-\frac{1}{2}\left(a_{K+1}^2+b_{K+1}^2\right)(\xi^2+\eta^2)\frac{T_{K-1}(\xi^2+\eta^2)}{T_K(\xi^2+\eta^2)}\right], \quad 1 \le K \le M-1. \tag{C-153}$$

Finally, the averages on a_M and b_M in (C-145) are given by

$$\frac{1}{2\pi}\iint da_M\, db_M \exp\left[-\frac{1}{2}(a_M^2+b_M^2)\left\{1+(\xi^2+\eta^2)\frac{T_{M-2}(\xi^2+\eta^2)}{T_{M-}(\xi^2+\eta^2)}\right\}\right]\frac{1}{T_{M-1}(\xi^2+\eta^2)}$$
$$= \frac{1}{T_M(\xi^2+\eta^2)}. \tag{C-154}$$

Equation (C-154) is the second-order CF of the RVs $\tilde{u}$ and $\tilde{v}$ in (C-144). Then by the procedure given in (C-44) through (C-46), we obtain

$$\begin{aligned} \mathrm{Prob}(\ell > T) &= P_F(\mu=1) = \mathrm{Prob}\left(|\tilde{u}+i\tilde{v}| > T/\sigma^2\right) \\ &= 1-\int_0^\infty dx\, J_1(x)\Big/T_M\left(\frac{\sigma^4}{T^2}x^2\right) \\ &= 1-\int_0^\infty dx\, J_1(x)\Big/T_M\left(\frac{x^2}{2(M-1)\lambda^2}\right), \end{aligned} \tag{C-155}$$

where normalized threshold

$$\lambda \equiv \frac{T}{\sqrt{2(M-1)}\,\sigma^2}. \tag{C-156}$$

The reason for the scaling is that

$$P_F(\mu=1) \sim \exp(-\lambda^2/2) \text{ as } M \to \infty. \tag{C-157}$$

To prove this, we use (C-151) to show that

$$\begin{aligned} T_M\left(\frac{x^2}{2(M-1)\lambda^2}\right) &= \sum_{K=0}^{[M/2]} \frac{(M-K)!}{(M-2K)!\,(M-1)^K}\frac{1}{K!}\left(\frac{x^2}{2\lambda^2}\right)^K \\ &\sim \sum_{K=0}^{\infty}\frac{1}{K!}\left(\frac{x^2}{2\lambda^2}\right)^K = \exp\left(\frac{x^2}{2\lambda^2}\right) \quad \text{as } M\to\infty. \end{aligned} \tag{C-158}$$

Therefore (C-155) yields

$$P_F(\mu=1) \sim 1-\int_0^\infty dx\, J_1(x)\exp\left(-\frac{x^2}{2\lambda^2}\right) = \exp(-\lambda^2/2) \text{ as } M\to\infty, \tag{C-159}$$

upon using reference 6, 6.631 5 and 8.352 1.

Now we will evaluate (C-155) in closed form by first determining the roots of the polynomial $T_M(y)$. We notice that RV $\tilde{u}$ in (C-144) is equal to RV ℓ/σ^2 in (C-87) for $m=0$; but, the CF of the RV ℓ/σ^2 is given by (C-94) and (C-93). Therefore the CF of RV $\tilde{u}$ in (C-144) is equal to

$$\prod_{k=1}^{M} \left(1 - i2\rho_k \xi\right)^{-1} = \prod_{k=1}^{[M/2]} \left(1 + 4\rho_k^2 \xi^2\right)^{-1}, \tag{C-160}$$

where

$$\rho_k \equiv \cos\left(\frac{k\pi}{M+1}\right), \quad 1 \le k \le M. \tag{C-161}$$

However, from (C-154), the CF of RV $\tilde{u}$ is given by $T_M^{-1}(\xi^2)$: Set $y=0$ in (C-145) and (C-154). Therefore

$$T_M(y) = \prod_{k=1}^{[M/2]} \left(1 + 4\rho_k^2 y\right). \tag{C-162}$$

In order to evaluate (C-155), we first expand $T_M^{-1}(y)$ in partial fractions:

$$T_M^{-1}(y) = \frac{1}{\prod_{k=1}^{[M/2]} \left(1 + 4\rho_k^2 y\right)} = \sum_{k=1}^{[M/2]} \frac{R_k}{1 + 4\rho_k^2 y}. \tag{C-163}$$

(Notice that $\sum_{k=1}^{[M/2]} R_k = 1$ since $T_M(0) = 1$.) The coefficients in (C-163) are given by

$$R_\ell = \prod_{\substack{k=1 \\ k \ne \ell}}^{[M/2]} \left(1 - \frac{\rho_k^2}{\rho_\ell^2}\right) = \prod_{\substack{k=1 \\ k \ne \ell}}^{[M/2]} \left(1 - \frac{\cos^2\left(\frac{k\pi}{M+1}\right)}{\cos^2\left(\frac{\ell\pi}{M+1}\right)}\right), \quad 1 \le \ell \le \left[\frac{M}{2}\right]. \tag{C-164}$$

Substituting (C-163) in (C-155), we obtain

$$\begin{aligned} P_F(\mu=1) &= 1 - \sum_{k=1}^{[M/2]} R_k \int_0^\infty dx\, J_1(x) \left[1 + \frac{2\rho_k^2}{(M-1)\lambda^2} x^2\right]^{-1} \\ &= 1 - \sum_{k=1}^{[M/2]} R_k \left[1 - \chi_k K_1(\chi_k)\right] = \sum_{k=1}^{[M/2]} R_k \chi_k K_1(\chi_k), \end{aligned} \tag{C-165}$$

where

$$\chi_k \equiv \frac{\sqrt{(M-1)/2}}{\rho_k} \lambda = \frac{\sqrt{(M-1)/2}}{\cos\left(\frac{k\pi}{M+1}\right)} \lambda, \quad 1 \le k \le \left[\frac{M}{2}\right]. \tag{C-166}$$

The integral in (C-165) was evaluated by means of reference 13, part II, 521. 6. The function K_1 in (C-165) is a modified Bessel function of the second kind of order one. Equation (C-165) is a closed-form expression of the false alarm probability for $\mu = 1$; the quantities $\{R_k\}$ are given by (C-164), and the quantities $\{\chi_k\}$ are given by (C-166). Table C-7 gives values of λ, as determined from (C-165), for $M = 25$ and $\mu = 1$.

Table C-7. Thresholds Required for Processor VII for M = 25 and μ = 1; Unknown Signal Frequency

P_F (μ = 1)	λ (P_F, M = 25)
10^{-1}	2. 16662
10^{-2}	3. 24388
10^{-3}	4. 16957
10^{-4}	5. 0227

Approximate Detection Probability

The RVs u and v in (C-135) are approximately Gaussian for large M, and we can concentrate on their second moments. We have already evaluated the mean and variance of u in (C-121) and (C-124). The mean of RV v is zero, as may be seen using (C-116). The mean of RV uv can be determined in a manner similar to that in (C-122) and (C-123); the result is found to be zero. The mean-square value of RV v is

$$\begin{aligned}
\overline{v^2} &= \frac{1}{(M-1)^2} \sum_{k,\ell=2}^{M} E\left\{\left(S_k C_{k-1} - C_k S_{k-1}\right)\left(S_\ell C_{\ell-1} - C_\ell S_{\ell-1}\right)\right\} \\
&= \frac{1}{(M-1)^2}\left[\sum_{k=2}^{M} E\left\{\left(S_k C_{k-1} - C_k S_{k-1}\right)\left(S_k C_{k-1} - C_k S_{k-1}\right)\right\}\right. \\
&\quad + \sum_{k=2}^{M-1} E\left\{\left(S_k C_{k-1} - C_k S_{k-1}\right)\left(S_{k+1} C_k - C_{k+1} S_k\right)\right\} \\
&\quad + \sum_{k=3}^{M} E\left\{\left(S_k C_{k-1} - C_k S_{k-1}\right)\left(S_{k-1} C_{k-2} - C_{k-1} S_{k-2}\right)\right\} \\
&\quad \left. + \sum_{\substack{k,\ell=2 \\ |k-\ell|\geq 2}}^{M} E\left\{\left(S_k C_{k-1} - C_k S_{k-1}\right)\left(S_\ell C_{\ell-1} - C_\ell S_{\ell-1}\right)\right\}\right].
\end{aligned} \tag{C-167}$$

The four averages in (C-167) are, respectively,

$$2\,\overline{C^2}\,\overline{S}^2,\ -\,\overline{S^2}\,\overline{C}^2,\ -\,\overline{S^2}\,\overline{C}^2,\ 0. \tag{C-168}$$

Therefore the variance of RV v is

$$\sigma_v^2 = \frac{2}{M-1}\,\overline{S^2}\left[\overline{C^2} - \overline{C}^2 + \frac{1}{M-1}\,\overline{C}^2\right]. \tag{C-169}$$

The joint PDF of RVs u and v is approximated according to

$$p(u,v) \cong \left(2\pi\,\sigma_u\sigma_v\right)^{-1}\exp\left[-\frac{(u-m_u)^2}{2\sigma_u^2} - \frac{v^2}{2\sigma_v^2}\right]. \tag{C-170}$$

The probability of detection, using (C-134), is

$$P_D = \mathrm{Prob}(\ell > T) = \mathrm{Prob}\left(\sqrt{u^2+v^2} > \frac{T}{(M-1)\sigma^{2\mu}}\right)$$

$$\cong \iint\limits_{\sqrt{u^2+v^2} > \frac{T}{(M-1)\sigma^{2\mu}}} du\,dv\,\left(2\pi\sigma_u\sigma_v\right)^{-1}\exp\left[-\frac{(u-m_u)^2}{2\sigma_u^2} - \frac{v^2}{2\sigma_v^2}\right] \tag{C-171}$$

$$= 1 - \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} dx\,\exp\left[-\frac{(x-b)^2}{2}\right]\left[2\Phi\left(\sqrt{c-dx^2}\right)-1\right] \equiv \tilde{P}_D,$$

where

$$a = \frac{T}{(M-1)\sigma^{2\mu}\sigma_u},\quad b = \frac{m_u}{\sigma_u},$$

$$c = \frac{T^2}{(M-1)^2\sigma^{4\mu}\sigma_v^2},\quad d = \frac{\sigma_u^2}{\sigma_v^2}. \tag{C-172}$$

For $m=0$, changing to polar coordinates in (C-171) and employing (C-130) and (C-169), there follows for the approximate false alarm probability

$$\tilde{P}_F = \exp\left(-\lambda^2/2\right), \tag{C-173}$$

where normalized threshold

$$\lambda \equiv \frac{T}{\sqrt{M-1}\,2^{\mu-\frac{1}{2}}\,\sigma^{2\mu}\,\Gamma(1+\mu)}. \tag{C-174}$$

which agrees with (C-142) and (C-156) for $\mu=0$ and $\mu=1$, respectively. Then the quantities (C-172) that are necessary for $\tilde{P}_D$ in (C-171) are expressible in terms of λ as

$$a = \frac{2^{\mu-\frac{1}{2}}\,\Gamma(1+\mu)}{(M-1)^{1/2}\,\sigma_u}\,\lambda\,,\quad b = \frac{m_u}{\sigma_u}\,,\quad c = a^2 d^2\,,\quad d = \frac{\sigma_u^2}{\sigma_v^2}\,. \tag{C-175}$$

The quantities m_u, σ_u^2, and σ_v^2 are themselves available in (C-121), (C-124), and (C-169), respectively.

Equations (C-171), (C-173), and (C-175) constitute the approximations to P_D and P_F for arbitrary μ. Threshold λ is determined from (C-173) for a specified $\tilde{P}_F$, and substituted in (C-171) to evaluate $\tilde{P}_D$. However, for $\mu=0$ or $\mu=1$, λ should be determined from (C-141) or (C-165), respectively, and substituted in (C-171) for evaluation of $\tilde{P}_D$.

Approximations for Small m

For small SNRs per sample, $m \ll 1$. The approximations in (C-130) are valid in this case, as is

$$\sigma_v \sim \left(\frac{2}{M-1}\right)^{1/2} 2^{\mu-1}\,\Gamma(1+\mu) \equiv \sigma_0\,. \tag{C-176}$$

Then (C-171) becomes

$$\begin{aligned} \tilde{P}_D &\sim \iint\limits_{\sqrt{u^2+v^2} > \frac{T}{(M-1)\sigma^{2\mu}}} du\,dv\,\left(2\pi\sigma_0^2\right)^{-1} \exp\left[-\frac{(u-m_u)^2+v^2}{2\sigma_0^2}\right] \\ &= Q\left(\frac{\Gamma^2\left(\frac{3+\mu}{2}\right)}{\Gamma(1+\mu)}\left(\frac{M-1}{2}\right)^{1/2} m^2\,,\ \lambda\right) \quad \text{as } m \to 0, \end{aligned} \tag{C-177}$$

where we changed to polar coordinates and used (C-8), (C-176), (C-174), and (C-130). The factor involving μ in (C-177) was encountered previously in (C-131) and was tabulated in table C-5; the comments made there are relevant here also. It is important to note the m^2-dependence in (C-177) versus the linear m-dependence in Processor II, (C-7), and in Processor V, (C-68). Thus, in addition to the suppression effect caused by the lack of knowledge of signal phase ψ_0, the phase-difference processor suffers further small-signal suppression, for unknown signal frequency, for all values of weighting constant μ in (C-133).

Appendix D

SIMULATION PROCEDURE AND PROGRAM

D.1 SIMULATION

In order to determine the ROCs for the processors in tables 1 and 2, 10,000 independent trials were conducted for each processor. A trial consisted of the generation of $2M$ independent zero-mean unit-variance Gaussian RVs. To keep computer time and storage at reasonable levels, several different mean values were simultaneously added (in different summation registers) to each of the RVs, which were then subjected to the various operations shown in tables 1 and 2. The number of threshold excursions of the resultant RVs were evaluated for a wide range of threshold settings. A plot of the relative number of threshold crossings for a particular nonzero-mean versus the relative number for a zero mean gave one curve of a ROC. A sample program for Processors I-VI is given in section D.2.

For a given added mean value, the number of excursions versus threshold settings are highly correlated RVs, resulting from the method described above for generating the ROCs. This means that if the simulated ROC is above (below) the true ROC at one value of P_F, it will very likely be above (below) the true value at other nearby values of P_F. The way we have chosen to control this effect is to take many trials, namely, 10,000. Thus the standard deviation of a particular probability evaluation is $(P(1-P)/10{,}000)^{1/2} \leq .005$, where P is the probability of the particular event.

In order to determine the correlation of the estimated ROC values, consider the following: The relative number of excursions above threshold A for N trials, $\{x_n\}_1^N$, is

$$r(A) = \frac{1}{N}\sum_{n=1}^{N} U(x_n - A), \tag{D-1}$$

where $U(y)$ equals 1 if $y>0$, and equals 0 if $y<0$. The mean of RV $r(A)$ is

$$\overline{r(A)} = \overline{U(x-A)} = 1 - P(A) \equiv Q(A), \tag{D-2}$$

where $P(A)$ is the probability that RV x remains less than A. If we keep the same realization of the RVs $\{x_n\}_1^N$ as in (D-1), but subject them instead to threshold B, yielding RV

$$r(B) = \frac{1}{N}\sum_{n=1}^{N} U(x_n - B), \tag{D-3}$$

the crosscorrelation coefficient of $r(A)$ and $r(B)$ readily follows as

$$\rho = \frac{Q(\max(A,B)) - Q(A)\,Q(B)}{\left[P(A)\,Q(A)\,P(B)\,Q(B)\right]^{1/2}}, \tag{D-4}$$

using the independence of x_n and x_m for $n \neq m$. So, for example, if $A = B$, (D-4) yields $\rho = 1$, as it must. However, if $A = 0$ and $B = 1$, for a zero-mean Gaussian RV x_n, there follows

$$P(A) = .5, \quad P(B) = .841, \quad \rho = .435. \tag{D-5}$$

Thus probability estimates as different as .500 and .841 are still rather highly correlated. In order to avoid overly distorted estimates of the ROCs, a large number of independent trials, N, must be used.

D.2 SAMPLE PROGRAM FOR PROCESSORS I THROUGH VI

```
      PARAMETER M= 25,BINS=1000,TRIALS=10000,NUMSNR=5
      DIMENSION P11(BINS),P12(BINS),P13(BINS),P14(BINS),P15(BINS)
     $,         P21(BINS),P22(BINS),P23(BINS),P24(BINS),P25(BINS)
     $,         P31(BINS),P32(BINS),P33(BINS),P34(BINS),P35(BINS)
     $,         P41(BINS),P42(BINS),P43(BINS),P44(BINS),P45(BINS)
     $,         P51(BINS),P52(BINS),P53(BINS),P54(BINS),P55(BINS)
     $,         P61(BINS),P62(BINS),P63(BINS),P64(BINS),P65(BINS)
      DIMENSION SA(NUMSNR),SB(NUMSNR),SC(NUMSNR),SD(NUMSNR),SE(NUMSNR),
     $SF(NUMSNR),A(6),B(6),Z(200),R(6),SG(NUMSNR)
      DIMENSION O1(BINS,NUMSNR),O2(BINS,NUMSNR),O3(BINS,NUMSNR),
     $          O4(BINS,NUMSNR),O5(BINS,NUMSNR),O6(BINS,NUMSNR)
      REAL L1(NUMSNR),L2(NUMSNR),L3(NUMSNR),L4(NUMSNR),L5(NUMSNR),
     $L6(NUMSNR),MAXM,MAXMSQ,MEAN,MSTART
      EQUIVALENCE (O1(1,1),P11),(O1(1,2),P12),(O1(1,3),P13)
      EQUIVALENCE (O1(1,4),P14),(O1(1,5),P15)
      EQUIVALENCE (O2(1,1),P21),(O2(1,2),P22),(O2(1,3),P23)
      EQUIVALENCE (O2(1,4),P24),(O2(1,5),P25)
      EQUIVALENCE (O3(1,1),P31),(O3(1,2),P32),(O3(1,3),P33)
      EQUIVALENCE (O3(1,4),P34),(O3(1,5),P35)
      EQUIVALENCE (O4(1,1),P41),(O4(1,2),P42),(O4(1,3),P43)
      EQUIVALENCE (O4(1,4),P44),(O4(1,5),P45)
      EQUIVALENCE (O5(1,1),P51),(O5(1,2),P52),(O5(1,3),P53)
      EQUIVALENCE (O5(1,4),P54),(O5(1,5),P55)
      EQUIVALENCE (O6(1,1),P61),(O6(1,2),P62),(O6(1,3),P63)
      EQUIVALENCE (O6(1,4),P64),(O6(1,5),P65)
      DEFINE F(I,K)=I*K+((1-SIGN(1,I*K))/2)*34359738367
      MSTART=.4
      DELTAM=.1
      K=5**15
      I=5281
      OOM=1./M
      OOMSQ=OOM**2
      OOT=1./TRIALS
```

```
      SQM=SQRT(OOM)
      MAXM=DELTAM*(NUMSNR-1)+MSTART
      MAXMSQ=MAXM**2
      A(1)=-4.*SQM
      B(1)=MAXM+4.*SQM
      A(2)=0.
      B(2)=MAXMSQ+2.*OOM+8.*SQM*SQRT(MAXMSQ+OOM)
      A(3)=2.-8.*SQM
      B(3)=2.+MAXMSQ+8.*SQM*SQRT(MAXMSQ+1.)
      A(4)=-2.*SQRT(2.)*SQM
      B(4)=1.
      A(5)=0.
      B(5)=1.
      A(6)=A(3)
      B(6)=B(3)
      DO 11 NP=1,6
11    R(NP)=BINS/(B(NP)-A(NP))
      DO 1 IT=1,TRIALS
      DO 9 J=1,NUMSNR
      SA(J)=0.
      SB(J)=0.
      SC(J)=0.
      SD(J)=0.
      SE(J)=0.
      SF(J)=0.
      SG(J)=0.
9     CONTINUE
      PSI=6.28318531/TRIALS*IT
      CP=COS(PSI)
      SP=SIN(PSI)
      DO 2 IM=1,M
      I=F(I,K)
      W=TINORM(FLOAT(I)/34359738367.,$3)
      GO TO 4
3     PRINT 5,
5     FORMAT(' PROBLEM')
4     I=F(I,K)
      V=TINORM(FLOAT(I)/34359738367.,$6)
      GO TO 7
6     PRINT 5,
7     DO 8 J=1,NUMSNR
      MEAN=DELTAM*(J-1)+MSTART
      IF(J.EQ.1) MEAN=0.
      WPM=W+MEAN
      X=WPM*CP-V*SP
      Y=V*CP+WPM*SP
      SA(J)=SA(J)+X
      SB(J)=SB(J)+Y
      XS=X*X
      YS=Y*Y
      SC(J)=SC(J)+XS
      SD(J)=SD(J)+YS
      T=SQRT(XS+YS)
      IF(J.EQ.1) GO TO 41
      SG(J)=SG(J)+LOG(BSSL(T*MEAN,2))
```

```
41    T=1./T
      SE(J)=SE(J)+X*T
      SF(J)=SF(J)+Y*T
8     CONTINUE
2     CONTINUE
      DO 10 J=1,NUMSNR
      L1(J)=OOM*(SA(J)*CP+SB(J)*SP)
      L2(J)=OOMSQ*(SA(J)**2+SB(J)**2)
      L3(J)=OOM*(SC(J)+SD(J))
      L4(J)=OOM*(SE(J)*CP+SF(J)*SP)
      L5(J)=OOMSQ*(SE(J)**2+SF(J)**2)
      IF(J.GT.1) GO TO 42
      L6(J)=L3(J)
      GO TO 10
42    MEAN=DELTAM*(J-1)+MSTART
      L6(J)=4.*OOM/MEAN**2*SG(J)
10    CONTINUE
C   CALCULATION OF PDF
      DO 12 J=1,NUMSNR
      N=(L1(J)-A(1))*R(1)
      N=MAX(N,1)
      N=MIN(N,BINS)
      O1(N,J)=O1(N,J)+OOT
      N=(L2(J)-A(2))*R(2)
      N=MAX(N,1)
      N=MIN(N,BINS)
      O2(N,J)=O2(N,J)+OOT
      N=(L3(J)-A(3))*R(3)
      N=MAX(N,1)
      N=MIN(N,BINS)
      O3(N,J)=O3(N,J)+OOT
      N=(L4(J)-A(4))*R(4)
      N=MAX(N,1)
      N=MIN(N,BINS)
      O4(N,J)=O4(N,J)+OOT
      N=(L5(J)-A(5))*R(5)
      N=MAX(N,1)
      N=MIN(N,BINS)
      O5(N,J)=O5(N,J)+OOT
      N=(L6(J)-A(6))*R(6)
      N=MAX(N,1)
      N=MIN(N,BINS)
      O6(N,J)=O6(N,J)+OOT
12    CONTINUE
1     CONTINUE
C   CALCULATION OF CDF
      DO 13 J=1,NUMSNR
      DO 13 IB=2,BINS
      O1(BINS+1-IB,J)=O1(BINS+1-IB,J)+O1(BINS+2-IB,J)
      O2(BINS+1-IB,J)=O2(BINS+1-IB,J)+O2(BINS+2-IB,J)
      O3(BINS+1-IB,J)=O3(BINS+1-IB,J)+O3(BINS+2-IB,J)
      O4(BINS+1-IB,J)=O4(BINS+1-IB,J)+O4(BINS+2-IB,J)
      O5(BINS+1-IB,J)=O5(BINS+1-IB,J)+O5(BINS+2-IB,J)
      O6(BINS+1-IB,J)=O6(BINS+1-IB,J)+O6(BINS+2-IB,J)
```

```
 13   CONTINUE
      CALL MODESG(Z,0)
      CALL SUBJEG(Z,0.,0.,1.,1.)
      CALL OBJCTG(Z,1150.,1000.,2850.,2700.)
      CALL LINESG(Z,BINS,P11,P12)
      CALL LINESG(Z,BINS,P11,P13)
      CALL LINESG(Z,BINS,P11,P14)
      CALL LINESG(Z,BINS,P11,P15)
      CALL SUBPLT
      CALL LINESG(Z,BINS,P21,P22)
      CALL LINESG(Z,BINS,P21,P23)
      CALL LINESG(Z,BINS,P21,P24)
      CALL LINESG(Z,BINS,P21,P25)
      CALL SUBPLT
      CALL LINESG(Z,BINS,P31,P32)
      CALL LINESG(Z,BINS,P31,P33)
      CALL LINESG(Z,BINS,P31,P34)
      CALL LINESG(Z,BINS,P31,P35)
      CALL SUBPLT
      CALL LINESG(Z,BINS,P41,P42)
      CALL LINESG(Z,BINS,P41,P43)
      CALL LINESG(Z,BINS,P41,P44)
      CALL LINESG(Z,BINS,P41,P45)
      CALL SUBPLT
      CALL LINESG(Z,BINS,P51,P52)
      CALL LINESG(Z,BINS,P51,P53)
      CALL LINESG(Z,BINS,P51,P54)
      CALL LINESG(Z,BINS,P51,P55)
      CALL SUBPLT
      CALL LINESG(Z,BINS,P61,P62)
      CALL LINESG(Z,BINS,P61,P63)
      CALL LINESG(Z,BINS,P61,P64)
      CALL LINESG(Z,BINS,P61,P65)
      CALL SUBPLT
      PFL=.1
      DO 21 IB=1,BINS
      K1=IB
 21   IF(P11(IB).LT.PFL) GO TO 22
 22   DO 23 IB=1,BINS
      K2=IB
 23   IF(P21(IB).LT.PFL) GO TO 24
 24   DO 25 IB=1,BINS
      K3=IB
 25   IF(P31(IB).LT.PFL) GO TO 26
 26   DO 27 IB=1,BINS
      K4=IB
 27   IF(P41(IB).LT.PFL) GO TO 28
 28   DO 29 IB=1,BINS
      K5=IB
 29   IF(P51(IB).LT.PFL) GO TO 30
 30   DO 31 IB=1,BINS
      K6=IB
 31   IF(P61(IB).LT.PFL) GO TO 32
 32   CALL SUBJEG(Z,0.,0.,PFL,1.)
      CALL LINESG(Z,BINS+1-K1,P11(K1),P12(K1))
      CALL LINESG(Z,BINS+1-K1,P11(K1),P13(K1))
```

```
      CALL LINESG(Z,BINS+1-K1,P11(K1),P14(K1))
      CALL LINESG(Z,BINS+1-K1,P11(K1),P15(K1))
      CALL SUBPFL
      CALL LINESG(Z,BINS+1-K2,P21(K2),P22(K2))
      CALL LINESG(Z,BINS+1-K2,P21(K2),P23(K2))
      CALL LINESG(Z,BINS+1-K2,P21(K2),P24(K2))
      CALL LINESG(Z,BINS+1-K2,P21(K2),P25(K2))
      CALL SUBPFL
      CALL LINESG(Z,BINS+1-K3,P31(K3),P32(K3))
      CALL LINESG(Z,BINS+1-K3,P31(K3),P33(K3))
      CALL LINESG(Z,BINS+1-K3,P31(K3),P34(K3))
      CALL LINESG(Z,BINS+1-K3,P31(K3),P35(K3))
      CALL SUBPFL
      CALL LINESG(Z,BINS+1-K4,P41(K4),P42(K4))
      CALL LINESG(Z,BINS+1-K4,P41(K4),P43(K4))
      CALL LINESG(Z,BINS+1-K4,P41(K4),P44(K4))
      CALL LINESG(Z,BINS+1-K4,P41(K4),P45(K4))
      CALL SUBPFL
      CALL LINESG(Z,BINS+1-K5,P51(K5),P52(K5))
      CALL LINESG(Z,BINS+1-K5,P51(K5),P53(K5))
      CALL LINESG(Z,BINS+1-K5,P51(K5),P54(K5))
      CALL LINESG(Z,BINS+1-K5,P51(K5),P55(K5))
      CALL SUBPFL
      CALL LINESG(Z,BINS+1-K6,P61(K6),P62(K6))
      CALL LINESG(Z,BINS+1-K6,P61(K6),P63(K6))
      CALL LINESG(Z,BINS+1-K6,P61(K6),P64(K6))
      CALL LINESG(Z,BINS+1-K6,P61(K6),P65(K6))
      CALL SUBPFL
      PRINT 33, (IB,P11(IB),P12(IB),P13(IB),P14(IB),P15(IB),IB=K1,BINS)
      PRINT 33, (IB,P21(IB),P22(IB),P23(IB),P24(IB),P25(IB),IB=K2,BINS)
      PRINT 33, (IB,P31(IB),P32(IB),P33(IB),P34(IB),P35(IB),IB=K3,BINS)
      PRINT 33, (IB,P41(IB),P42(IB),P43(IB),P44(IB),P45(IB),IB=K4,BINS)
      PRINT 33, (IB,P51(IB),P52(IB),P53(IB),P54(IB),P55(IB),IB=K5,BINS)
      PRINT 33, (IB,P61(IB),P62(IB),P63(IB),P64(IB),P65(IB),IB=K6,BINS)
33    FORMAT(I10,5E20.8)
      CALL EXITG(Z)
      SUBROUTINE SUBPLT
      CALL LINESG(Z,0,0.,0.)
      CALL LINESG(Z,1,0.,1.)
      CALL LINESG(Z,1,1.,1.)
      CALL LINESG(Z,1,1.,0.)
      CALL LINESG(Z,1,0.,0.)
      CALL LINESG(Z,1,1.,1.)
      CALL SETSMG(Z,30,.5)
      DO 14 IL=1,9
      CALL LINESG(Z,0,.1*IL,0.)
      CALL LINESG(Z,1,.1*IL,1.)
      CALL LINESG(Z,0,0.,.1*IL)
14    CALL LINESG(Z,1,1.,.1*IL)
      CALL PAGEG(Z,0,2,1)
      CALL SETSMG(Z,30,1.)
      RETURN
      SUBROUTINE SUBPFL
      CALL LINESG(Z,0,0.,0.)
      CALL LINESG(Z,1,0.,1.)
```

```
      CALL LINESG(Z,1.,.1,1.)
      CALL LINESG(Z,1.,.1,0.)
      CALL LINESG(Z,1,0.,0.)
      CALL LINESG(Z,1.,.1,.1)
      CALL SETSMG(Z,30,.5)
      DO 15 IL=1,9
      CALL LINESG(Z,0.,.01*IL,0.)
      CALL LINESG(Z,1.,.0.*IL,1.)
      CALL LINESG(Z,0,0.,.1*IL)
15    CALL LINESG(Z,1.,.1,.1*IL)
      CALL PAGEG(Z,0,2,1)
      CALL SETSMG(Z,30,1.)
      RETURN
      END
```

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